FINAL: 120 Minutes

Last Name: Solutions

First Name: ______

RIN: ______

Section: ______

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
You **MUST** show **CORRECT** work (even for multiple choice) to receive full credit.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
200	30	30	30	30	30	350

1 Circle at most one answer per question. 10 points for each correct answer.

(1) True or false: $\exists C > 0 : \forall n \geq 1 : 10n^2 \geq C(n^3 + n)$. $\rightarrow 10n^2 \in \Omega(n^3)$

$10n^2 \in o(n^3)$

False

B

- [A] True.
- [B] False.
- [C] It depends on C.
- [D] It depends on n.
- [E] None of the above.

(2) True or false: $\forall n \geq 1 : \exists C > 0 : 10n^2 \geq C(n^3 + n)$.

A

- [A] True.
- [B] False.
- [C] It depends on C.
- [D] It depends on n.
- [E] None of the above.

pick $C = \frac{1}{n}$

$\frac{1}{n}(n^3+n) = n^2+1 \leq n^2+n^2 \leq 2n^2 \leq 10n^2$

$\therefore$ true.

(3) What is the set $\mathcal{A}$, recursively defined on the right.

$1 \rightarrow {2 \atop 0} \rightarrow {3 \atop -1} \rightarrow {4 \atop -2} \rightarrow \cdots \} \mathbb{Z}$.

B

- [A] $\mathbb{N}$.
- [B] $\mathbb{Z}$.
- [C] $\mathbb{Q}$.
- [D] $\mathbb{R}$.
- [E] None of the above.

(1) $1 \in \mathcal{A}$.
(2) $x, y \in \mathcal{A} \rightarrow x + y \in \mathcal{A}$ AND $x - y \in \mathcal{A}$.
(3) Nothing else is in $\mathcal{A}$.

(4) $T_1 = 1$ and $T_n = T_{n-1} + \sqrt{n}$ for $n > 1$. Estimate T_{100}?

C

SHOW WORK

- [A] 7.
- [B] 70.
- [C] 700.
- [D] 7000.
- [E] 70000.

$T_1 = 1$
$T_2 = 1 + \sqrt{2}$
$T_3 = 1 + \sqrt{2} + \sqrt{3}$
$\vdots$
$T_n = \sum_{i=1}^{n} \sqrt{i} \sim \int_1^n dx\sqrt{x} \sim \frac{2}{3} n^{3/2} = \frac{2}{3}(100)^{3/2}$
$= 1000 \cdot \frac{2}{3}$
≈ 666.

(5) Compute the sum $S = \sum_{i=1}^{4} \sum_{j=1}^{4} ij^2$.

$\sum_{i=1}^{4} i \sum_{j=1}^{4} j^2$

B

SHOW WORK.

- [A] 290.
- [B] 300.
- [C] 310.
- [D] 320.
- [E] None of the above.

$\frac{1}{2} \cdot 4(4+1) \cdot \frac{1}{6} 4 \cdot (4+1)(2 \cdot 4 + 1)$

$= 10 \cdot \frac{20}{6} \cdot 9 = 5 \cdot 20 \cdot 3 = 300$

(6) The sum $S(n) = \sum_{i=1}^{n}(i^2 + i)$. Which is true?

order of summand = 2

nestings = 1

$\therefore \Theta(n^3)$.

D.

- $\boxed{\text{A}}$ $S(n) \in \Theta(n)$.
- $\boxed{\text{B}}$ $S(n) \in \Theta(n^2)$.
- $\boxed{\text{C}}$ $S(n) \in \Theta(n^2 \log n)$.
- $\boxed{\text{D}}$ $S(n) \in \Theta(n^3)$.
- $\boxed{\text{E}}$ None of the above.

(7) The hour hand on a clock points to 1 o'clock. After 2200^{2200} hours, where will the hour hand be pointing?

$12 \overline{)2200}$ 0183 r4

$\therefore\ 2200 \equiv 4 \mod 12$

$4^2 \equiv 4 \mod 12$

$\therefore\ 4^k \equiv 4 \mod 12$

$\therefore\ (2200)^{2200} \equiv 4^{2200} \equiv 4 \mod 12$.

$\therefore\ 1 + (2250)^{2250} \equiv 5 \mod 12 \to$ 5 o'clock.

B

SHOW WORK

- $\boxed{\text{A}}$ 2 o'clock.
- $\boxed{\text{B}}$ 5 o'clock.
- $\boxed{\text{C}}$ 8 o'clock.
- $\boxed{\text{D}}$ 11 o'clock.
- $\boxed{\text{E}}$ None of the above.

(8) In a graph, the only two vertices with odd degree are u and v. Must there be a path from u to v.

If u, v are in different connected components $\to$ each is a graph with 1 odd-degree vertex. Not Possible Handshaking theorem $\to$ even # of odd-degrees. $\therefore$ u, v in the same connected component.

A

- $\boxed{\text{A}}$ Yes, always.
- $\boxed{\text{B}}$ No, never.
- $\boxed{\text{C}}$ It is possible or not, depending on the number of edges.
- $\boxed{\text{D}}$ It is possible or not, depending on the number of vertices.
- $\boxed{\text{E}}$ Such a graph cannot exist.

(9) A class has 10 students. How many different debate teams with 5 kids are possible?

$\binom{10}{5} = \frac{10!}{5!5!}$

D.

- $\boxed{\text{A}}$ 10^5.
- $\boxed{\text{B}}$ 5^{10}.
- $\boxed{\text{C}}$ $10 \times 9 \times 8 \times 7 \times 6$.
- $\boxed{\text{D}}$ $\binom{10}{5}$.
- $\boxed{\text{E}}$ None of the above.

(10) In how many ways can you distribute ten \$1 bills among three children aged 1,2,3 so that each child gets an amount of money that is at least their age?

After distributing the minimums $\to$ 4 left to distribute.

1 0 | 0 0 | 0 0 0 |

1 2 3

0 0 1 0 0 1 binary sequences with 2 1's of length 6.

$\binom{6}{2} = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2} = 15$

E

SHOW WORK

- $\boxed{\text{A}}$ 8
- $\boxed{\text{B}}$ 9
- $\boxed{\text{C}}$ 10
- $\boxed{\text{D}}$ 11
- $\boxed{\text{E}}$ None of the above.

(11) How many 6-bit strings have 00 as a substring. [*Hint*: Let $Q_n = \#n$-bit strings ....]

[A] 21.

[B] 32.

[C] 43.

[D] 64.

[E] None of the above.

C

SHOW WORK

Q_n = # n-bit strings with 00. The ~~...~~ strings start 1, 01 or 00:

$Q_0 = 0$ $Q_1 = 0$ $Q_2 = 1$

$1 \to Q_{n-1}$

$01 \to Q_{n-2}$

$00 \to 2^{n-2}$

$Q_n = Q_{n-1} + Q_{n-2} + 2^{n-2}$

n	0	1	2	3	4	5	6
Q_n	0	0	1	3	8	19	$27+2^4=43$.

(12) Roll 3 dice. What are the chances of exactly 2 ones?

[A] 5/72.

[B] 6/72.

[C] 7/72.

[D] 8/72.

[E] None of the above.

A

Binomial. # trials = 3

prob success = $\frac{1}{6}$

$P[2 \text{ successes}] = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) = 3 \cdot \frac{1}{36} \cdot \frac{5}{6} = \frac{5}{12 \cdot 6} = \frac{5}{72}.$

(13) 60 students are split into FOCS (20 boys, 10 girls) and ALGO (10 boys, 20 girls). A random student is picked and it is a girl. What are the chances this student is in FOCS?

[A] 1/3.

[B] 1/4.

[C] 1/5.

[D] 1/6.

[E] None of the above.

A

SHOW WORK.

$P[\text{girl}] = \frac{30}{60} = \frac{1}{2}$

$P[\text{FOCS}/\text{girl}] = \frac{P[\text{FOCS} \cap \text{girl}]}{P[\text{girl}]} = \frac{10/60}{1/2}$

$= \frac{1}{6} \times 2 = \frac{1}{3}$

(14) $\mathbb{E}[\mathbf{X}] = 2, \mathbb{E}[\mathbf{Y}] = 3$. What is $\mathbb{E}[2\mathbf{X} + 3\mathbf{Y}]$?

[A] 10.

[B] 11.

[C] 13.

[D] 15.

[E] None of the above, or not enough information given.

C

Linearity of expectation

$2 \cdot E[x] + 3 \cdot E[y] = 2 \cdot 2 + 3 \cdot 3$

$= 4 + 9 = 13.$

(15) $\mathbb{E}[\mathbf{X}] = 2, \mathbb{E}[\mathbf{Y}] = 3$. What is $\mathbb{E}[\mathbf{X}^2 + 3\mathbf{Y}]$?

[A] 10.

[B] 11.

[C] 13.

[D] 15.

[E] None of the above, or not enough information given.

E

linearity of expectation.

$E[x^2] + 3E[y] = E[x^2] + 9$ ← can't say

could be anything

(16) Flip a fair coin until 1 or more heads ***or*** 2 or more tails. What is the expected number of flips?

A

- [A] 1.5.
- [B] 2.5.
- [C] 3.5
- [D] 4.5.
- [E] None of the above.

H $\frac{1}{2}$

T → H $\frac{1}{4}$

T → T $\frac{1}{4}$

flips	1	2
Prob	$\frac{1}{2}$	$\frac{1}{2}$

$E[\#\text{ flips}] = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 2$

$= \underline{1.5}$

(17) Flip a fair coin until 1 or more heads ***and*** 2 or more tails. What is the expected number of flips?

D

Show work

- [A] 1.5.
- [B] 2.5.
- [C] 3.5
- [D] 4.5.
- [E] None of the above.

H ← waiting for 2T → $1 + \frac{2}{1/2} = 1 + 4 = 5$

T, H ← waiting for T → $2 + \frac{1}{1/2} = 4$

T, T ← waiting for H → $2 + 1/1/2 = 4$

Total Expectation

$E[\#\text{ flips}] =$

$= \frac{1}{2}\cdot 5 + \frac{1}{4}\cdot 4 + \frac{1}{4}\cdot 4$

$= \frac{5}{2} + 1 + 1$

$= \underline{4.5}$

(18) Toss 10 balls randomly into 10 bins. What is the expected number of bins with exactly 1 ball.

C

Show work.

- [A] $9^8/10^8$.
- [B] $9^8/10^9$.
- [C] $9^9/10^8$.
- [D] $9^9/10^9$.
- [E] None of the above.

$P[\text{bin 1 has exactly 1 ball}] = \binom{10}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^9$ ← Binomial

$= \left(\frac{9}{10}\right)^9$

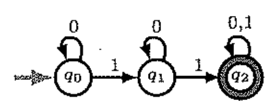
$X_i = \begin{cases} 1 & \text{bin } i \text{ has exactly 1 ball} \\ 0 & \text{otherwise} \end{cases}$

$X = X_1 + X_2 + \cdots + X_{10} \Rightarrow E[X] = 10 \cdot E[X_1] = 10\cdot\left(\frac{9}{10}\right)^9 = \frac{9^9}{10^8}$

(19) What is the computing problem solved by the DFA on the right?

C

- [A] Strings with an even number of 1s.
- [B] Strings with two 1s.
- [C] Strings with at least two 1s.
- [D] Strings with more than two 1s.
- [E] None of the above.



Two 1s get you to the accept state where you stay → at least 2 ones

(20) $\mathcal{L}_A$ is reducible to $\mathcal{L}_B$, that is $\mathcal{L}_A \leq_{\mathrm{R}} \mathcal{L}_B$. We know $\mathcal{L}_B$ is decidable. Therefore:

C

- [A] $\mathcal{L}_A$ must be finite. ✗
- [B] $\mathcal{L}_A$ must be infinite. ✗
- [C] $\mathcal{L}_A$ must be decidable. ✓
- [D] $\mathcal{L}_A$ must be undecidable. ✗
- [E] None of the above. ✗

$\mathcal{L}_A$ is easier ∴ $\mathcal{L}_A$ is decidable.

↑ can be finite or infinite.

2 **Pick any 11 distinct numbers from $\{1, 2, \ldots, 15\}$. Prove that three are consecutive.**

For example, if you pick $\{1, 2, 4, 5, 6, 7, 8, 10, 11, 14, 15\}$ then 4,5,6 are consecutive.

Define the 5 bins

$\lfloor 1\ 2\ 3 \rfloor\ \lfloor 4\ 5\ 6 \rfloor\ \lfloor 7\ 8\ 9 \rfloor\ \lfloor 10\ 11\ 12 \rfloor\ \lfloor 13\ 14\ 15 \rfloor$

Each number you pick falls in some bin.

If each bin has at most 2 numbers $\rightarrow$ you picked at most 5×2 numbers $= 10$

$\therefore$ some bin must have at least 3 numbers } PIGEONHOLE

~~That bin~~ Those 3 numbers are consecutive.

50% Some understanding

80% Basic Idea

100% Basically correct proof.

3 $\mathcal{L} = \{111, 11111\}^*$. **What strings are in $\mathcal{L}$. Prove your answer.**

$\mathcal{L} = \{\varepsilon, 111, 11111, 1^6, 1^8, 1^9, 1^{10}, 1^{11}, \ldots\}$

$1^3 \cdot 1^3 \quad 1^3 \cdot 1^5 \quad 1^3 \cdot 1^3 \cdot 1^3 \quad \ldots$

50% TINKER.

It looks like 1^k is in $\mathcal{L}$ for $k \geq 8$.

So then $\mathcal{L} = \{\varepsilon, 1^3, 1^5, 1^6, 1^8, 1^9, 1^{10}, 1^{11}, 1^{12}, \ldots\}$.

80% get correct guess and Basic idea and induction proof are correct.

We need to prove this. INDUCTION

Easiest is leaping Induction because $P(n) \to P(n+3)$

Claim $1^n \in \mathcal{L}$ for $n \geq 8$.

Base Cases.

$$\left.\begin{aligned} 1^8 &= 1^3 \cdot 1^5 \\ 1^9 &= 1^3 \cdot 1^3 \cdot 1^3 \\ 1^{10} &= 1^5 \cdot 1^5 \end{aligned}\right\} \text{3 base cases.}$$

100% Correct induction proof.

Induction ~~Assume~~ Show $P(n) \to P(n+3)$

Assume $1^n \in \mathcal{L}$.

Prove $1^{n+3} \in \mathcal{L}$

$1^n = a_1 \cdot a_2 \cdot \ldots a_\ell$ where $a_i \in \mathcal{L}$.

$\to \underbrace{a_1 \cdot a_2 \cdot \ldots \cdot a_\ell}_{1^{n+1}} \cdot 1^3 \in \mathcal{L}$

$\therefore$ $P(n+3)$ is true and $P(n) \to P(n+3)$.

$\therefore$ $1^n \in \mathcal{L}$ for $n \geq 8$. $\blacksquare$

Clearly

4 **A "Diagonal" binomial sum. Prove by induction:** $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{n}$, **for** $m, n \geq 0$.

Fix any $m \geq 0$ ⟵ Must give argument for general m.

$$P(n):\ \sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{n}$$

Proof By Induction

Base Case. $P(0)$ claims $\underbrace{\sum_{k=0}^{0} \binom{m+k}{k}}_{1 = \binom{m}{0}} = \binom{m+1}{0} = 1$ } clearly true.

Induction Prove $P(n) \rightarrow P(n+1)$

Assume $P(n)$ $\quad \sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{n}$

Prove $P(n+1)$ $\quad \sum_{k=0}^{n+1}\binom{m+k}{k} = \binom{m+n+2}{n+1}$

$$\sum_{k=0}^{n+1}\binom{m+k}{k} = \sum_{k=0}^{n}\binom{m+k}{k} + \binom{m+n+1}{n+1}$$

$$= \binom{m+n+1}{n} + \binom{m+n+1}{n+1} \longleftarrow \text{INDUCTION HYPOTHESIS}$$

$$= \frac{(m+n+1)!}{n!\,(m+1)!} + \frac{(m+n+1)!}{(n+1)!\,m!}$$

$$= \frac{(m+n+1)!}{n!\,m!}\left[\frac{1}{m+1} + \frac{1}{n+1}\right]$$

$$= \frac{(m+n+1)!}{n!\,m!}\left[\frac{m+n+2}{(m+1)(n+1)}\right]$$

$$= \frac{(m+n+2)!}{(n+1)!\,(m+1)!}$$

$$= \binom{m+n+2}{n+1} \checkmark$$

50% Base case and Correct Induction Structure

80% Basically correct idea to prove $P(n) \Rightarrow P(n+1)$

Cannot assert $P(n+1)$ as if true

100% Basically Correct Proof.

5 Expected number of runs.

A biased coin with probability 1/3 of heads is flipped 10 times. In a run, all consecutive flips are the same, for example HHTHTTTTHH has five runs. Compute the expected number of runs.

Indicator random variables.

$$\text{Let } X_i = \begin{cases} 1 & \text{if flip } i \text{ starts a new run} \longrightarrow \text{flip } i \neq \text{flip } i-1 \\ 0 & \text{if flip } i \text{ does not start a new run} \rightarrow \text{flip } i = \text{flip } i-1 \end{cases}$$

$$P[X_i = 0] = \underset{HH}{\left(\tfrac{1}{3}\right)^2} + \underset{TT}{\left(\tfrac{2}{3}\right)^2} \qquad P[X_i = 1] = 2 \cdot \tfrac{1}{3} \cdot \tfrac{2}{3} = \tfrac{4}{9}$$

$X_1 = 1 \leftarrow$ First flip always starts a run.

$\#$ of runs $= X_1 + X_2 + \cdots + X_{10}$.

$$E[\# \text{ runs}] = E[X_1] + E[X_2] + \cdots + E[X_{10}] = 1 + \cancel{9} \times \frac{4}{\cancel{9}} = \boxed{5}$$

Build Up Expectation Suppose the first flip is either H or T
Let $X = \#$ runs.

$$\tfrac{1}{3}\ H \qquad \tfrac{2}{3}\ T$$

$$E[X] = \tfrac{1}{3} E[X \mid H] + \tfrac{2}{3} E[X \mid T]$$

Let $H(n)$ = Expect # runs of n flips stating with H

Let $T(n)$ = Expected # runs of n flips starting with T

We want $E[X] = \frac{1}{3} H(n) + \frac{2}{3} T(n)$.

Note $H(1) = 1$, $T(1) = 1$, $Q(1) = 1$

$H(n)$: $H \xrightarrow{\frac{2}{3}} HT \leftarrow 1 + T(n-1)$; $H \xrightarrow{\frac{1}{3}} HH \leftarrow H(n-1)$

$$\longrightarrow H(n) = \tfrac{1}{3} H(n-1) + \tfrac{2}{3}(1 + T(n-1))$$

$T(n)$: $T \xrightarrow{2/3} TT \quad T(n-1)$; $T \xrightarrow{\frac{1}{3}} TH \quad 1 + H(n-1)$

$$\longrightarrow T(n) = \tfrac{1}{3}(1 + H(n-1)) + \tfrac{2}{3} T(n-1)$$

$$\underbrace{\tfrac{1}{3} H(n) + \tfrac{2}{3} T(n)}_{Q(n)} = \underbrace{\tfrac{1}{3} H(n-1) + \tfrac{2}{3} T(n-1)}_{Q(n-1)} + \tfrac{4}{9} = Q(n-1) + 4/9$$

n	1	2	3	4	5	6	7	8	9	10
$H(n)$	1	5/3								
$T(n)$	1	4/3								
$Q(n)$	1	13/9	17/9	21/9	25/9	29/9	33/9	37/9	41/9	45/9

$= \boxed{5}$

$1 + \frac{4}{9}$

$Q(n) = 1 + (n-1) \cdot \frac{4}{9}$.

50% Understood Problem.

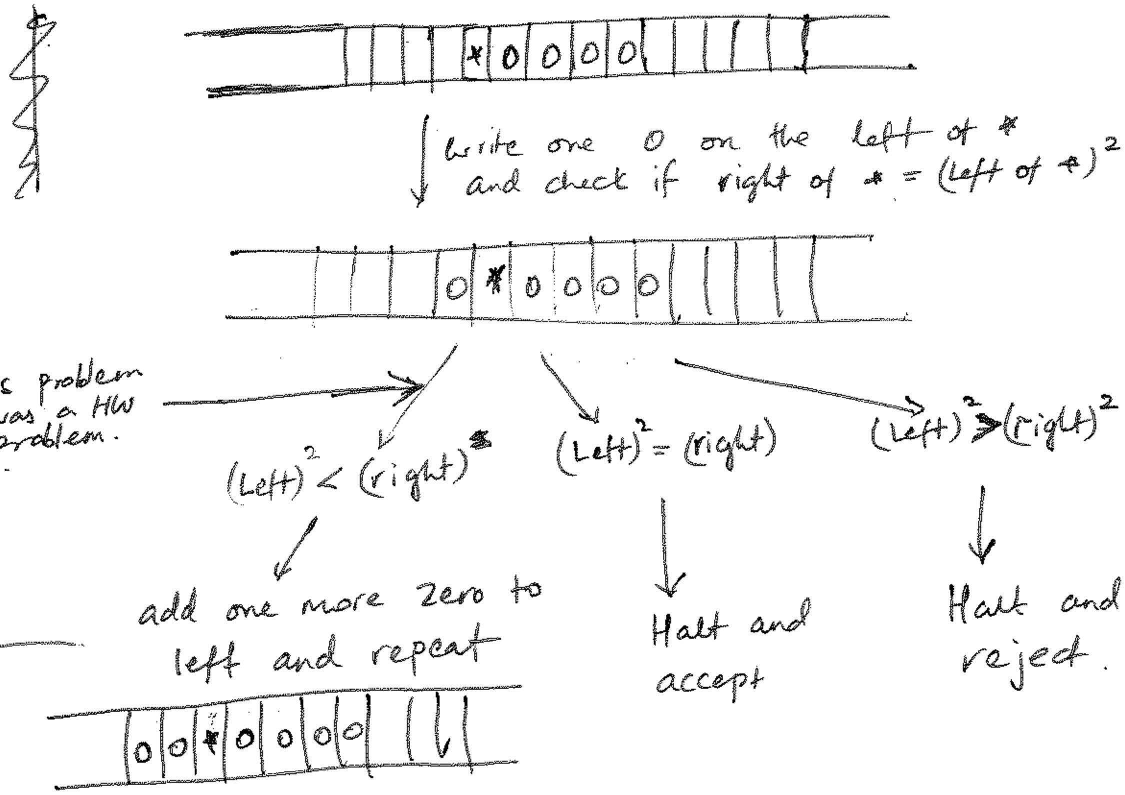
80% Generally one of the correct ideas. Can get here if took $p = 1/2$ not $p = \frac{1}{3}$

100% Generally correct derivation with all correct ideas.

6 **Give a sketch (high-level pseudocode) for a Turing Machine to Solves $\mathcal{L} = \{0^{\bullet n^2} | n \geq 0\}$.**

Many ways to solve. Simplest approach is to try $1^2\ 2^2\ 3^2 \ldots n^2$.

General Idea.



50% Understood Problem and challenge.

80% Some general algorithm that works.
Could use multiple marking strategies.

100% Correct TM algorithm