CSCI 2200 RPI
Foundations of Computer Science (FOCS) 2015

MIDTERM: 90 Minutes

Last Name: Solutions

First Name: ____________

RIN: ____________

Section: ____________

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
50	50	50	50	50	250

1 Circle one answer per question. 10 points for each correct answer.

(a) Compute the sum $\sum_{i=0}^{10}(i+1+2^{i+1})$

A 4150.

B 5160.

C 4149.

D 4160.

D.

$\sum i + \sum 1 + 2 \cdot \sum 2^i$

$\frac{10 \cdot 11}{2} + 11 + 2 \cdot (2^{11}-1)$

$55 + 11 + 2 \cdot (2048-1) = 66 + 4094 = 4160$

(b) Give a formula for the sum $S(n) = \sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)$

A $S(n) = n^3$

B $S(n) = n^2(n+1)$

C $S(n) = \frac{1}{2}n(n+1)$

D $S(n) = i^3 + j^3$

B

$\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} i + \sum_{i=1}^{n}\sum_{j=1}^{n} j$

$= 2\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} i}_{ni} = 2n\sum_{i=1}^{n} i = 2n\frac{n}{2}(n+1) = n^2(n+1)$

(c) Let $S(n) = \sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)$. Then,

A $S(n) \in \Theta(n^3)$

B $S(n) \in \Theta(n^2 \log n)$

C $S(n) \in \Theta(n)$

D $S(n) \in \Theta(n^4)$

A

$S(n) = n^2(n+1) = \Theta(n^3)$

(d) Compute the remainder when 2015^{2015} is divided by 3? *[Hint:* $2015 \equiv -1 \pmod 3$*.]*

A $r = 1$

B $r = 2$

C $r = 3$

D $r = 7$

B

$2015 \equiv -1 \mod 3$

$2015^{2015} \equiv (-1)^{2015}$

$\equiv -1 \mod 3$

$\equiv 2 \mod 3$

(e) A friendship network has 7 people and each person has 4 friends. How many edges (friendship links) are there in this friendship network?

A 14 edges

B 15 edges

C Not enough information to determine the number of edges

D This friendship network cannot possibly exist

A.

$2E = 4 \cdot 7 = 28$

$\underline{E = 14}$

2 Computing and Proving a Sum

Give a formula for the sum $S(n) = \sum_{i=1}^{2n} (-1)^i i$. Prove your answer.

$$\underbrace{-1+2}_{1} \; \underbrace{-3+4}_{1} \; \underbrace{-5+6}_{1} \; \cdots \; \underbrace{-(2n-1)+2n}_{1} = n. \quad \leftarrow \text{conjecture}$$

$P(n)$: $\sum_{i=1}^{2n} (-1)^i i = n$. Claim $P(n)$ $\forall n \geq 1$.

Proof by induction

Base case. $n=1$ $\sum_{i=1}^{2\cdot 1} (-1)^i i = 1-2 = 1 = 1. = n$ ✓

Induction Assume $P(n)$: $\sum_{i=1}^{2n} (-1)^i i = n$.

Prove $P(n+1)$: $\sum_{i=1}^{2(n+1)} (-1)^i i = (n+1)$

$$\sum_{i=1}^{2n+2} (-1)^i i = \underbrace{\sum_{i=1}^{2n} (-1)^i i}_{n \text{ by I.H.}} + \underbrace{(-1)}_{-1}^{2n+1} (2n+1) + \underbrace{(-1)}_{+1}^{2n+2} (2n+2)$$

$$= n - (2n+1) + (2n+2)$$

$$= n+1 \quad \boxed{}$$

By induction $P(n)$ is T $\forall$ $n \geq 1$.

3 Greatest Common Divisor (GCD)

For an integer d and integers m, n, suppose that $\gcd(d, m) = 1$ and $d|mn$. Prove that $d|n$.

$\gcd(d, m) = 1 \rightarrow xd + ym = 1$ [Bezout].

$\rightarrow xnd + ymn = n$ [multiply by n].

$mn = \alpha d \rightarrow xnd + y\alpha d = n$

$\rightarrow n = (xn + y\alpha)\, d$

$\therefore$ $d|n$. $\square$

4 Regular Graphs

A graph is r-regular if every node has degree r. Let n be the number of nodes in the graph.

(a) Prove that if n and r are both odd, then there is no r-regular graph on n nodes.

$2E = \Sigma$ degrees $= nr$

n, r both odd mean nr is odd

If such a graph exists, then

$2E =$ odd

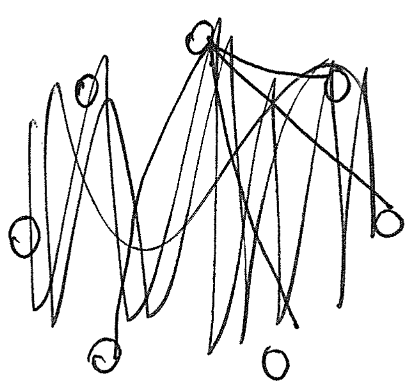
thats a contradiction $2E$ is even.

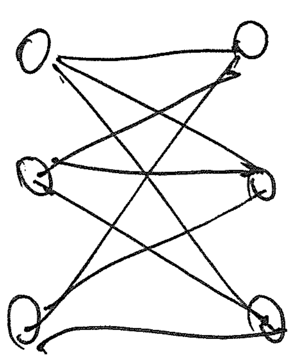
$\therefore$ such a graph cannot exist.

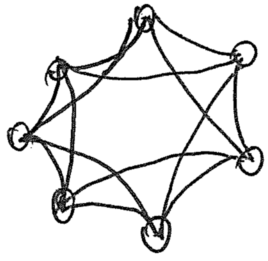
(b) Draw examples of r-regular graphs in these two cases:

(i) $n = 7$; $r = 4$

(ii) $n = 6$; $r = 3$



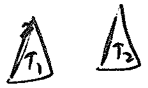
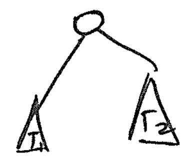




5 Rooted Full Binary Trees

Give the recursive definition of rooted full binary trees. (State your base cases and constructor rules.)

① o ∈ RFBT

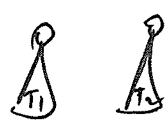
②  ∈ RFBT →  ∈ RFBT

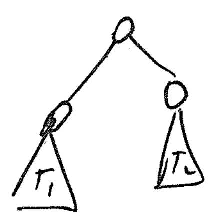
③ Nothing else is a RFBT.

Prove that the number of nodes in any rooted full binary tree is odd.

Proof By structural Induction

①. Base Case o has 1 node which is odd.

② suppose  have n_1, n_2 nodes, odd

Consider the tree created  with $1+n_1+n_2$ nodes.

Since $n_1 = 2k_1+1$, $n_2 = 2k_2+1$

$n = 1 + 2k_1 + 2k_2 + 2$

$= 1 + 2(k_1+k_2+1)$

Which is odd $\boxed{}$

By structural induction, every RFBT has an odd # of nodes.