MIDTERM: 90 Minutes

Last Name: Solutions

First Name: ________

RIN: ________

Section: ________

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You **MUST** show your work, even on multiple choice questions, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
150	20	20	20	20	20	250

1 Circle one answer per question. 10 points for each correct answer.

(1) All digits of $n > 1$ are 1 (e.g. 111 or 11111). What is the remainder when this number is divided by 4?

D

- [A] 0.
- [B] 1.
- [C] 2.
- [D] 3.
- [E] It depends on the number of digits.

$10^2 \equiv 0 \pmod 4$ (100 is div by 4)

$\rightarrow 10^i \equiv 0 \pmod 4$ $i \geq 2$.

$\therefore 1 + 10 + 10^2 + 10^3 + \cdots 10^n = \underbrace{1111\ldots1}_{n\text{-ones}}$

$\equiv 1 + 10 + 0 + 0 + \cdots + 0 \pmod 4$

$\equiv 3 \pmod 4$

(2) What is the last digit of n (i.e. remainder when divided by 10), where $n = 2017^{2017} - 17^{2017}$?

A

- [A] 0
- [B] 1
- [C] 3
- [D] 7
- [E] 9

$2017 \equiv 17 \pmod{10}$

$\rightarrow 2017^{2017} \equiv 17^{2017} \pmod{10}$

$\therefore \underline{2017^{2017} - 17^{2017} \text{ is div by } 10}$

(3) [**Hard**] Which theorem below is true (n is a natural number)?

B

- [A] n^2 is divisible by 10 if and only if n is divisible by 4. $n=4, n^2=16$ (counterexample)
- [B] n^2 is divisible by 8 if and only if n is divisible by 4. ✓ $n^2 = 2^3 \times k$ $n = 2^a m \rightarrow n^2\ 2^{2a} m^2$ $\rightarrow 2a \geq 3 \rightarrow a \geq 3/2 \rightarrow \underline{a \geq 2}$ n div by 4.
- [C] n^2 is divisible by 4 if and only if n is divisible by 4. $n=2$ $n^2=4$ (counter-example)
- [D] n^2 is divisible by 2 if and only if n is divisible by 4. $n=2$ n^2 is div by 2 (counterexample)
- [E] None of the above.

(4) Compute the sum $S = \sum_{i=1}^{2} \sum_{j=1}^{2} (i + j)$

D

- [A] $S = 6$.
- [B] $S = 8$.
- [C] $S = 10$.
- [D] $S = 12$.
- [E] $S = 14$.

j

$i+j$	1	2
1	2	3
2	3	4

i

} sum = 12

(5) How many subsets of $\{a, b, c, d, e, f, g\}$ contain a or g?

A 32

B 64

C 96

D 108

E 128

C

total subsets $= 2^7$

do not contain a AND g $\Rightarrow$ Subset of $\{b, c, d, e, f\}$

$\rightarrow 2^5$ of those.

contain a OR g $= 2^{\cancel{7}} - 2^5 = 128 - 32$

$= 96$

(6) *Estimate* the value of $2^1 \times 2^2 \times \cdots \times 2^{10} = \prod_{i=1}^{10} 2^i$.

A 3.6×10^{10}.

B 3.6×10^{12}.

C 3.6×10^{14}.

D 3.6×10^{16}.

E 3.6×10^{18}.

D

$2^{1+2+\cdots+10} = 2^{55}$ $\frac{10 \times 11}{2}$.

$2^{55} = 2^5 \times 2^{50} = 2^5 \times (2^{10})^5$ $2^{10} \approx 1000 = 10^3$

$32 \times 10^{3 \cdot 5}$ lower estimate because $2^{10} > 10^3$

$\approx 3.2 \times 10^{16}$.

(7) Which is the correct asymptotic order relationship between $f(n) = 2^n$ and $g(n) = \sum_{i=0}^{n} 2^i$

A $f \in \Theta(g)$.

B $f \in \omega(g)$.

C $f \in o(g)$.

D None of the above.

A

$\sum_{i=1}^{n} 2^i = 2^{n+1} - 1$

$\frac{2^{n+1} - 1}{2^n} \rightarrow 2$ = constant $\therefore f \in \Theta(g)$

(8) $\{a_1, a_2, a_3, a_4, a_5\} = \{1, 4, 9, 16, 25\}$ and $\{b_1, b_2, b_3, b_4, b_5\} = \{1, -1, 1, -1, 1\}$. Compute $\sum_{i=1}^{5} \sum_{j=1}^{5} a_i b_j$.

A 51.

B 53.

C 55.

D 57.

E 59.

C

$\sum_{i=1}^{5} \sum_{j=1}^{5} a_i b_j$ nested sum. compute inner sum first

$\sum_{j=1}^{5} a_i b_j = a_i \sum_{j=1}^{5} b_j$ [constant rule]

$= a_i [\underbrace{1 - 1 + 1 - 1 + 1}_{1}] = a_i \Rightarrow \sum_{i=1}^{5} a_i = \underbrace{1+4+9+16+25}_{55.}$

(9) A friendship network (simple graph) has vertices having degree sequence $\delta = [4, 4, 4, 2, 2]$. How many edges (friendship links) are in this friendship network?

must be connected to all vertices

$\therefore$ all vertices have degree ≥ 3

$\to$ not possible.

E

$\boxed{A}$ 7 edges

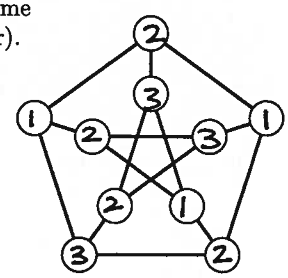
$\boxed{B}$ 8 edges

$\boxed{C}$ 9 edges

$\boxed{D}$ Not enough information to determine the number of edges

$\boxed{E}$ This friendship network cannot possibly exist

(10) You wish to color the Petersen graph so that linked vertices do not get the same color. What is the minimum number of colors that you need (the chromatic number).



B

½ credit C.

$\boxed{A}$ 2

$\boxed{B}$ 3

$\boxed{C}$ 4

$\boxed{D}$ 5

$\boxed{E}$ 6

(11) How many 5-letter strings have no two consecutive letters the same (letters are $a, b, \ldots, z$).

$\boxed{A}$ $26 \times 25 \times 24 \times 23 \times 22$.

$\boxed{B}$ 26^5

$\boxed{C}$ $\binom{30}{4}$.

$\boxed{D}$ 26×25^4.

$\boxed{E}$ None of the above.

D

String: $x_1\, x_2\, x_3\, x_4\, x_5$

$\uparrow$ 26 choices, $\uparrow$ 25 choices (cannot match x_1), $\uparrow$ 25 (cannot match x_2), $\uparrow$ 25, $\uparrow$ 25

$\Big\}\ 26 \times 25^4$ (product rule).

(12) Evaluate $\binom{8}{3} - \binom{8}{5}$

$\boxed{A}$ 0.

$\boxed{B}$ 1.

$\boxed{C}$ 2.

$\boxed{D}$ 3.

$\boxed{E}$ 4.

A

$$\binom{n}{k} = \binom{n}{n-k} \to \binom{8}{3} = \binom{8}{5} \therefore \text{ answer} = 0.$$

$$\frac{8!}{5!\,3!} = \frac{8!}{3!\,5!}$$

(13) A 10×10 *binary* array B has i 1's in each row. How many choices are there for B?

A 100^i.

B $\binom{100}{10 \times i}$.

C $\binom{10}{i}^{10}$.

C

D $\frac{100!}{i!}$.

E $\frac{100!}{(100-i)!}$.

each row has i 1s $\}$ $\binom{10}{i}$ ways. for each row

row length = 10

$\therefore$ $\binom{10}{i} \times \binom{10}{i} \times \cdots \binom{10}{i} = \binom{10}{i}^{10}$ [product rule]

(14) What is the coefficient of x^3 in the expansion of $(2-3x)^5$?

A 360.

B -360.

C 720.

D -720.

E 1080.

E

Binomial Theorem

$2^2 \cdot (-3x)^3 \cdot \binom{5}{3}$

4 -27 10. $= -270 \times 4 = -1080$

(15) How many integer solutions for z_1, z_2, z_3 satisfy $0 \leq z_1 \leq z_2 \leq z_3 \leq 10$?

A $\binom{10}{3} = 120$.

B $\binom{13}{3} = 286$.

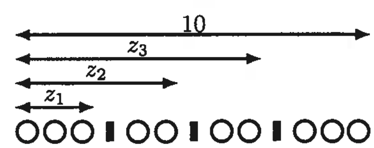
B

C $10 \times 9 \times 8 = 720$.

D $10^3 = 1000$.

E None of the above.

Hint: Here is a picture for the case $z_1 = 3, z_2 = 5, z_3 = 7$.



each integer solution $\Leftrightarrow$ length 13 binary sequence with 3 1's.

$\therefore$ $\binom{13}{3}$.

2 Common Divisors Divide the GCD

Prove that every common divisor of m and n divides $\gcd(m, n)$.

$$\gcd(m,n) = mx + ny \longleftarrow \text{Bezout}$$

any common divisor divides the RHS because it divides m and n.

$\therefore$ it divides the LHS $\square$

50% understood problem and make basic progress

80% reasonable attempt at proof

100% proof

3 Prove that $\log_{10} 9$ is an irrational number.

Proof by contradiction

Assume $\log_{10} 9$ is rational

$\therefore \log_{10} 9 = \frac{a}{b}$

$\rightarrow 9 = 10^{a/b}$

$\rightarrow 9^b = 10^a$

$\rightarrow \boxed{3^{2b} = 2^a 5^a}$

This means that some number has two different prime factorizations: 3^{2b} and $2^a 5^a$ which contradicts the fundamental theorem of arithmetic.

$\square$

50% contradiction

80% made progress

100% got to a contradiction

4 **Prove by *induction*:** $1+2+\cdots+n=\sum_{i=1}^{n} i \le n^2$.

(Note: There are many ways to prove this. You are required to prove it by *induction*.)

$P(n): \sum_{i=1}^{n} i \le n^2$.

Proof by induction

Base Case $P(1): \sum_{i=1}^{1} i = 1 \le 1^2$ ✓

Induction: Assume $P(n): \sum_{i=1}^{n} i \le n^2$

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n+1$$

$$\le n^2 + n+1 \quad \text{(induction hypothesis)}$$

$$\le n^2 + 2n + 1 \quad (n \ge 0 \therefore n \le 2n)$$

$$= (n+1)^2$$

$\therefore P(n+1)$ is T

By induction $P(n)$ is T for $n \ge 1$.

50% correct induction infrastructure

80% Correct Induction Step

100% Correct proof

5 A recursively defined set of numbers.

The set $\mathcal{A}$ is recursively defined as shown.
(By default, nothing else is in $\mathcal{A}$ – minimality.)

Prove that every number in $\mathcal{A}$ is a multiple of 3.

① $3 \in \mathcal{A}$.
② $x, y \in \mathcal{A} \to x + y \in \mathcal{A}$;
$x, y \in \mathcal{A} \to x - y \in \mathcal{A}$.

Proof by structural induction.

Base Case: 3 is a multiple of 3.

Induction: [Prove constructor preserves property].

x, y multiples of 3 $\rightarrow$ $x = 3k$
$y = 3m$

Constructor (i) $x + y = 3(k+m)$ is a multiple of 3 ✓

Constructor (ii) $x - y = 3(k-m)$ is a multiple of 3 ✓

50% correct structural induction infrastructure

80% correct induction step

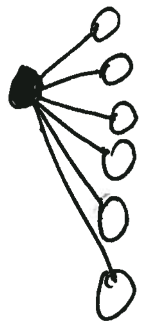
100% correct proof.

6 4-friendship-cliques and 3-wars.

Prove: among any 9 people, there are either 4 mutual friends (a 4-clique) or 3 mutual enemies (a 3-war). (You may assume that among any 6 people, either 3 are mutual friends or 3 are mutual enemies.)

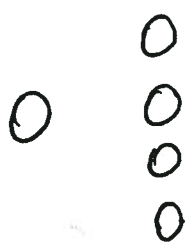
Three cases

i) Someone has 6 or more friends.



among those 6 friends is either a
3 war [done]
or a 3-clique.
Combine the 3-clique with the original friend to get a 4 clique [done].

(ii) Someone has fewer than 5 friends → they have at least 4 enemies



If any two are enemies, combine with the original to get a 3-war [done]
If all are mutual friends → 4-clique [done]

[iii] No one has 6 or more friends AND no one has fewer than 5 friends → Everyone has 5 friends.

9 people have 5 friends

Sum of degrees = $9 \times 5 = 45$ which is odd.

THIS CASE CANNOT HAPPEN [Handshaking theorem]

this problem is from the HW.

50% understood problem and made basic progress

80% understood the cases

100% correct step (iii) AND well written proof.