QUIZ 2: 60 Minutes

Last Name: Solutions

First Name: ____________

RIN: ____________

Section: ____________

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

10 Questions
Circle at most one answer per question.
10 points for each correct answer

Total
100

1. How many guests do you need at a party to *guarantee* that two will be born on the same day of the week?

pigeonhole: 8 guarantees
with 7: MTWTFSS are all possible

D

- A 3
- B 5
- C 7
- D 8
- E Not possible

2. How many guests do you need at a party to *guarantee* that two will be born on a Monday?

Everyone could be born on tuesday

E

- A 3
- B 5
- C 7
- D 8
- E Not possible

3. How many numbers in the set $\{1, 2, 3, \ldots, 1000\}$ are divisible by 2 *or* 3.

$500 = \lfloor \frac{1000}{2} \rfloor$ divisible by 2
$333 = \lfloor \frac{1000}{3} \rfloor$ divisible by 3
$166 = \lfloor \frac{1000}{6} \rfloor$ divisible by 6

Inclusion Exclusion:
$500 + 333 - 166 = 667$

C

- A 657
- B 660
- C 667
- D 830
- E 833

4. How many different words can you get by rearranging the letters of the word BOOKKEEPER?

$\frac{10!}{3!2!2!1!1!1!}$

3 : E
2 : O
2 : K
1 : B
1 : P
1 : R

B

- A $10!$
- B $\frac{10!}{2!\times 2!\times 3!}$
- C $\binom{10}{6}$
- D 6^{10}
- E 10^6

5. You have 11 players and must form two teams of 5 for a practice match. How many different practice matches are possible. (Be careful! **TINKER**: for example, try 3 players forming two teams of 1)?

$\frac{1}{2}\binom{11}{5}\cdot\binom{6}{5} = \frac{1}{\cancel{2}} \frac{11\times \cancel{10}\times 9\times \cancel{8}\times 7\times \cancel{6}}{\cancel{5\cdot 4\cdot 3\cdot 2}\cdot 1} \times \cancel{6} = 11\times 9\times 7\times 4 \times 2$

A vs B
B vs A } same practice match

99
$\times 7$
693
$\times 2$
1386

A

- A 1386
- B 1388
- C 1390
- D 2772
- E 2774

6. You randomly pick an 8-bit sequence (independent bits and each bit is 1 with probability $\frac{1}{2}$). What is the probability that the sequence starts and ends in 1?

$\boxed{A}$ $\frac{1}{2}$

$\boxed{B}$ $\frac{1}{4}$

$\boxed{C}$ $\frac{1}{8}$

$\boxed{D}$ $\frac{1}{16}$

$\boxed{E}$ $\frac{1}{32}$

B

$P[b_1=1 \wedge b_2=1] = P[b_1=1]P[b_2=1] = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

7. A box contains 10 coins. 9 are *fair* and 1 has *two heads*. You pick a coin at random. You toss it three times. What is the probability of tossing three heads (HHH)?

$\boxed{A}$ $\frac{1}{8}$

$\boxed{B}$ $\frac{15}{80}$

$\boxed{C}$ $\frac{16}{80}$

$\boxed{D}$ $\frac{17}{80}$

$\boxed{E}$ $\frac{18}{80}$

D

Total Probability

$P[HHH] = P[HHH \mid fair]\,P[fair] + P[HHH \mid 2\text{-heads}]\,P[2\text{-heads}].$

$= \frac{1}{8}\cdot\frac{9}{10} + 1\cdot\frac{1}{10}$

$= \frac{9}{80} + \frac{8}{80} = \frac{17}{80}.$

8. A box contains 10 coins. 9 are *fair* and 1 has two heads. You pick a coin at random. You toss your coin three times and get HHH. What is the probability that the coin you picked is fair?

$\boxed{A}$ $\frac{9}{10}$

$\boxed{B}$ $\frac{8}{17}$

$\boxed{C}$ $\frac{9}{17}$

$\boxed{D}$ $\frac{10}{17}$

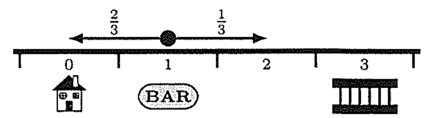
$\boxed{E}$ $\frac{11}{17}$

C

$P[fair \mid HHH] = \frac{P[fair \wedge HHH]}{P[HHH]} = \frac{P[HHH \mid fair]\,P[fair]}{P[HHH]}$ ← from 7

$= \frac{\frac{1}{8}\cdot\frac{9}{10}}{\frac{17}{80}} = \frac{\frac{9}{80}}{\frac{17}{80}} = \frac{9}{17}.$

9. A drunk leaves the bar (at position 1), and takes independent steps: left (L) with probability $\frac{2}{3}$ or right (R) with probability $\frac{1}{3}$. What is the probability the drunk reaches home (at position 0) before reaching the lockup (at position 3)?



$\boxed{A}$ $\frac{1}{2}$

$\boxed{B}$ $\frac{2}{3}$

$\boxed{C}$ $\frac{4}{5}$

$\boxed{D}$ $\frac{5}{6}$

$\boxed{E}$ $\frac{6}{7}$

E

Total Probability: $P[H] = \underbrace{P[H|L]}_{1}\,\underbrace{P[L]}_{\frac{2}{3}} + \underbrace{P[H|RL]}_{P[H]}\,\underbrace{P[RL]}_{\frac{1}{3}\times\frac{2}{3}} + \underbrace{P[H|RR]\,P[RR]}_{0}$

$= \frac{2}{3} + \frac{2}{9}\cdot P[H]$

$\rightarrow \quad \frac{7}{9}P[H] = \frac{2}{3} \rightarrow \quad P[H] = \frac{2}{3}\times\frac{9}{7} = \frac{6}{7}$

Sum outcome Probabilities. outcomes leading to home are $(RL)^i L$.

$P[(RL)^i L] = \left(\frac{2}{3}\times\frac{1}{3}\right)^i \frac{2}{3} = \frac{2}{3}\cdot\left(\frac{2}{9}\right)^i$

$P[H] = \sum_{i=0}^{\infty} \frac{2}{3}\cdot\left(\frac{2}{9}\right)^i = \frac{2}{3}\cdot\frac{1}{1-\frac{2}{9}} = \frac{2}{3}\cdot\frac{1}{\frac{7}{9}} = \frac{6}{7}$

10. You toss 4 independent fair dice. What is the probability that you roll exactly *one* 2 and *one* 4?

$\boxed{A}$ $\frac{3}{27}$

$\boxed{B}$ $\frac{4}{27}$

$\boxed{C}$ $\frac{5}{27}$

$\boxed{D}$ $\frac{6}{27}$

$\boxed{E}$ $\frac{7}{27}$

B

outcomes = 6^4

4 positions for 2

3 positions for 4 given 2.

remaining 2 position chosen in 4^2 ways

$\therefore \quad \frac{4\cdot 3\cdot 4^2}{6^4} = \frac{3\times 2^6}{3^4\cdot 2^4} = \frac{4}{27}.$

Multinomial: $\frac{4!}{1!1!2!}\;\underset{P[2]}{\frac{1}{6}}\cdot\underset{P[4]}{\frac{1}{6}}\cdot\underset{P[\sim(2\text{ or }4)]}{\frac{4}{6}} = \frac{4}{27}.$