QUIZ 3: 60 Minutes

Last Name: Solutions

First Name: ______________________

RIN: ______________________

Section: ______________________

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer

Total
100

1. A random variable **X** has PDF shown on the right. Compute $\mathbb{E}[\mathbf{X}]$ (expectation) and $\sigma^2(\mathbf{X})$ (variance).

$\mathbf{X}$	-2	-1	0	1	2
$P_{\mathbf{X}}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{3}{10}$

[A] $\mathbb{E}[\mathbf{X}] = 0$ $\sigma^2(\mathbf{X}) = 2$

[B] $\mathbb{E}[\mathbf{X}] = 0.6$ $\sigma^2(\mathbf{X}) = 1$

[C] $\mathbb{E}[\mathbf{X}] = 0.6$ $\sigma^2(\mathbf{X}) = 1.64$

[D] $\mathbb{E}[\mathbf{X}] = 1$ $\sigma^2(\mathbf{X}) = 1.8$

[E] $\mathbb{E}[\mathbf{X}] = 0.5$ $\sigma^2(\mathbf{X}) = 1.65$

C

$E[X] = \frac{-2}{10} - \frac{1}{10} + \frac{3}{10} + \frac{6}{10} = \frac{6}{10} = \underline{0.6}$.

$E[X^2] = \frac{4}{10} + \frac{1}{10} + \frac{3}{10} + \frac{12}{10} = \frac{20}{10} = 2$

$\sigma^2 = E[x^2] - E[x]^2 = 2 - (0.6)^2 = 2 - 0.36 = \underline{1.64}$.

2. **X** and **Y** are rolls of two independent fair dice and $\mathbf{Z} = \mathbf{X} + 2\mathbf{Y}$. Compute $\mathbb{E}[\mathbf{Z}]$ (expectation).

[A] 6

[B] 7

[C] $9\frac{1}{2}$

[D] $10\frac{1}{2}$

[E] 14

D.

$E[Z] = E[X] + 2\,E[Y]$

$= 3\frac{1}{2} + 2 \cdot 3\frac{1}{2} = \underline{10\frac{1}{2}}$.

3. **X** and **Y** are rolls of two independent fair dice and $\mathbf{Z} = \mathbf{X} + 2\mathbf{Y}$. Compute $\sigma^2(\mathbf{Z})$ (variance). (The variance of a *single* die roll is 35/12.)

[A] 70/12

[B] 105/12

[C] 140/12

[D] 175/12

[E] 210/12

D.

$\sigma^2(z) = \sigma^2(x) + \sigma^2(2y)$ ← independence.

$= \frac{35}{12} + 4 \cdot \sigma^2(y)$ ← $4 = 2^2$ (square constants).

$= 5 \cdot \frac{35}{12} = \underline{\frac{175}{12}}$.

4. For a random variable **X**, what does the standard deviation $\sigma(\mathbf{X})$ measure?

[A] The average value of **X** you will observe if you ran the experiment many times.

[B] The number of times you run the experiment (on average) before you observe the value $\mathbb{E}[\mathbf{X}]$.

[C] The size of the deviation between the observed value of **X** and the expected value $\mathbb{E}[\mathbf{X}]$.

[D] The probability that **X** will be larger than its expected value $\mathbb{E}[\mathbf{X}]$.

[E] The number of possible values of **X**.

C

5. You toss two coins. If you get HH for the tosses, you roll 12 dice and *count* the number of sixes as **X**. If you do not get HH for the tosses, you roll 6 dice and *count* the number sixes as **X**. Compute $\mathbb{E}[\mathbf{X}]$, the expected number of sixes rolled.

A

[A] 1.25

[B] 1.5

[C] 1.75

[D] 2

[E] 2.5

Total Expectation

$$E[X] = \underbrace{E[X \mid HH]}_{\frac{12}{6}} \underbrace{P[HH]}_{1/4} + \underbrace{E[X \mid \overline{HH}]}_{\frac{6}{6}} \underbrace{P[\overline{HH}]}_{3/4}$$

sum of 12 Bernoullis with $p = 1/6$

sum of 6 Bernoullis with $p = 1/6$

$$= 2 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{5}{4} = \underline{1.25}$$

6. **[Hard]** A Martian couple continues to have children until they have 2 males *in a row.* On Mars, *males are half as likely as females.* Assume children are *independent.* Let **X** be the number of children this couple will have. Compute $\mathbb{E}[\mathbf{X}]$, the expected number of children this couple will have.

D

[A] 9

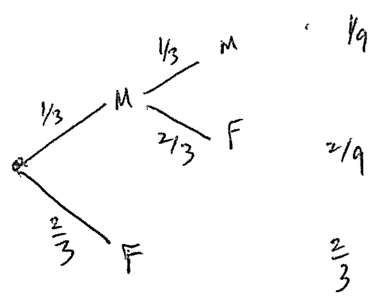
[B] 10

[C] 11

[D] 12

[E] 13

$P[\text{male}] = p = \frac{1}{3}$



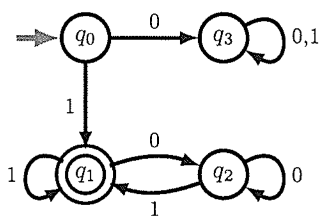
$$E[X] = \underbrace{E[X \mid MM]}_{2} \underbrace{P[MM]}_{1/9} + \underbrace{E[X \mid MF]}_{2+E[X]} \underbrace{P[MF]}_{2/9} + \underbrace{E[X \mid F]}_{1+E[X]} \underbrace{P[F]}_{6/9}$$

$$= \frac{2}{9} + (2 + E[X])\frac{2}{9} + (1 + E[X])\frac{6}{9}$$

$$= \frac{2}{9} + \frac{4}{9} + \frac{6}{9} + \frac{8}{9} E[X].$$

$$\rightarrow \quad E[X]\left(1 - \frac{8}{9}\right) = \frac{12}{9} \rightarrow \frac{E[X]}{9} = \frac{12}{9} \rightarrow \boxed{E[X] = 12}$$

7. The DFA on the right solves a computing problem defined by its (YES)-set (the language it accepts). What is a regular expression for this computing problem? [Recall $*$ is the wildcard.]



[A] $\{0,1\}^* \;=\; *$

[B] $\{1\}\bullet\{0,1\}^*\bullet\{1\} \;=\; 1*1$

[C] $\{1\}\bullet\{\{0,1\}^*\bullet 1\}^* \;=\; 1\{*1\}^*$

[D] $\{1\}^* \;=\; 1^*$

[E] $\{1\}\bullet\{01\}^* \;=\; 1(01)^*$

C

YES-SET = {1, 1*1}.
= 1{*1}*

8. What is the computing problem $\mathcal{L}$ solved by the CFG on the right? (What is the language generated?)

$S \to \varepsilon \mid 0 \mid 1 \mid SS$

[A] $\mathcal{L} = \{\text{all finite binary strings}\} = \Sigma^*$.

[B] $\mathcal{L} = \{\text{all finite binary strings of even length}\}$.

[C] $\mathcal{L} = \{\text{all finite binary strings of odd length}\}$.

[D] $\mathcal{L} = \{\text{all finite binary strings with an equal number of 0's and 1's}\}$.

[E] $\mathcal{L} = \{\text{all finite binary strings which contain 01 as a substring}\}$.

A

S: → SS → SSS ...
ε 0 1 00 01 10 11
all possible strings.

9. Which of the following is *countable*:

(I) Integers, $\mathbb{Z}$ (II) Valid C^{++} programs (III) The prime numbers

finite binary strings.

[A] I, II, III.

[B] only I, II.

[C] only I, III.

[D] only II, III.

[E] only I.

A.

10. rank deterministic finite automata (DFA), context free grammars (CFG), which are related to pushdown automata, and Turing Machines (TM) in order of how powerful they are. (For example, $DFA > CFG$ if DFAs can solve more problems that CFGs; $DFA = CFG$ if DFAs and CFGs can solve the same problems; $DFA < CFG$ if DFAs can solve fewer problems that CFGs.

[A] $DFA > CFG > TM$

[B] $DFA = CFG > TM$

[C] $DFA = CFG = TM$

[D] $DFA = CFG < TM$

[E] $DFA < CFG < TM$

E