QUIZ 1: 60 Minutes

Last Name: Solutions

First Name: ____________

RIN: ____________

Section: ____________

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer.

You **MUST** show work to get full credit.

When in doubt, TINKER.

Total
200

1. $\sqrt{2}$ is what kind of number?

$\boxed{A}$ A natural number.

$\boxed{B}$ A rational number.

$\boxed{C}$ An irrational number.

$\boxed{D}$ An integer.

$\boxed{E}$ None of the above.

2. The set $S = \{n \mid n = (k-1)(-1)^k,\ \text{where } k \in \mathbb{N}\}$. Which of these sets could be S?

$\boxed{A}$ $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$

$\boxed{B}$ $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$

$\boxed{C}$ $\{0, 1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \dots\}$

$\boxed{D}$ $\{1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \dots\}$

$\boxed{E}$ $\{0, -1, 2, -3, 4, -5, 6, -7, 8, -9, 10, \dots\}$

$S = \{0, 1, -2, 3, \dots$

3. A and B are sets. Which answer is another way to represent $\overline{A \cap B}$.

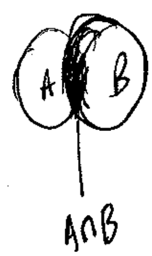
$\boxed{A}$ $A \cup B$.

$\boxed{B}$ $A \cap B$.

$\boxed{C}$ $\overline{A} \cup \overline{B}$.

$\boxed{D}$ $\overline{A} \cap \overline{B}$.

$\boxed{E}$ None of the above.



$\overline{A \cap B}$ is everything not in $A \cap B$, so it is everything outside A OR outside B, that is $\overline{A} \cup \overline{B}$

4. An integer $n \in \mathbb{Z}$ has an even square, that is n^2 is even. Which claim is true?

$\boxed{A}$ n is odd.

$\boxed{B}$ n is positive.

$\boxed{C}$ n^2 is divisible by 4.

$\boxed{D}$ n is divisible by 4.

$\boxed{E}$ None of the above claims are true.

n^2 is even $\rightarrow$ n is even

$n = 2k \rightarrow n^2 = 4k^2$

div by 4

5. How many rows are there in the truth table of the compound proposition $((p \to q) \vee (p \to r)) \to (q \to r)$?

$\boxed{A}$ 2.

$\boxed{B}$ 4.

$\boxed{C}$ 8.

$\boxed{D}$ 12.

$\boxed{E}$ 16.

3 variables $\rightarrow$ $2 \times 2 \times 2 = 8$.

6. On your car's back bumper is a sticker that says "Honk if you love FOCS." Joe was behind you and honked. Later, Sue was behind you and didn't honk. What would be a valid inference?

- [A] Joe loves FOCS. We don't know about Sue.
- [B] Sue loves FOCS. We don't know about Joe
- [C] Joe does not love FOCS. We don't know about Sue.
- [D] Sue does not love FOCS. We don't know about Joe
- [E] Joe loves FOCS and Sue does not love FOCS.

Joe honked → he may be excited

Sue did not honk → can't love FOCS

Joe honked: $p \to q$ q is T   can't infer p

Sue honked: $p \to q$ $\neg q$ so can infer $\neg p$.

7. For $x, y \in \mathbb{N} = \{1, 2, 3, \ldots\}$, determine T or F for the proposition $\forall y : (\exists x : x^2 = y)$.

- [A] Can't be done because p is not a valid proposition which is either T or F.
- [B] It depends on x.
- [C] It depends on y.
- [D] F.
- [E] T.

if $y = 2$

$\exists x : x^2 = 2 \to x = \sqrt{2} \notin \mathbb{N}$.

$\therefore$ not true for all y.

8. What method of proof did we use to prove that $\sqrt{2} \notin \mathbb{Q}$?

- [A] Direct proof
- [B] Contraposition proof.
- [C] Proof by induction.
- [D] Proof by contradiction.
- [E] None of the above.

9. What method would you use to *prove* that $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = (\frac{1}{2}n(n+1))^2$ for *all* $n \geq 1$?

- [A] Direct proof
- [B] Contraposition proof.
- [C] Show that the formula is true for $n = 1$ up to $n = 1000$.
- [D] Proof by induction.
- [E] Proof by contradiction.

10. You must prove $P(n)$ for $n \geq 3$. You proved $P(n) \to P(n+3)$ for $n \geq 3$. What base cases do you need?

- [A] $P(1)$
- [B] $P(3)$
- [C] $P(1)$, $P(2)$ and $P(3)$
- [D] $P(3)$, $P(4)$ and $P(5)$
- [E] None of the above.

$\boxed{P(3)}$ $\boxed{P(4)}$ $\boxed{P(5)}$ $P(6)$ $P(7)$ $P(8)$ . . .

Base Cases.

11. For $x, y \in \mathbb{N}$, which statement is a contradiction (cannot possibly be true)?

[A] $x^2 < y$. $x = 2 \quad y = 10$

[B] $x^2 = y/2$ $x = 2 \quad y = 8$

[C] $x^2 - y^2 \le 1$ $x = 2 \quad y = 2$

[D] $x^2 + y^2 \le 1$ $x^2 \geq 1 \quad y^2 \geq 1 \quad \therefore x^2 + y^2 \geq 2$ so D can't be true. → contradiction.

[E] None of the above. That is, each statement above can be true for specific choices of x, y.

12. Which gives a valid way to prove the implication $p \to q$.

prove $p = T$ $q = F$ can't happen.

[A] Assume p is F and show that q must be F. ✗

[B] Assume q is T and show that p must be T. ✗

[C] Assume p is T and show that q must be F. ✗

[D] Assume p is T and q is F and derive a contradiction. ✓

[E] None of the above.

13. What is the difference between using Induction versus Strong Induction to prove $P(n)$ for $n \geq 1$?

[A] The base cases are different.

[B] Induction is usually easier than Strong Induction.

[C] In Induction you prove $P(n+1)$. In Strong Induction you prove $P(n+2)$.

[D] In Induction you assume $P(n)$. In Strong Induction you assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$.

[E] There is no difference between the two methods.

14. Compute the value of $(1 - \frac{1}{2}) \times (1 - \frac{1}{3}) \times (1 - \frac{1}{4}) \times (1 - \frac{1}{5}) \times \cdots \times (1 - \frac{1}{100})$.

$\frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \cdots \frac{99}{100} = \frac{1}{100}$

[A] 1/5

[B] 1/10

[C] 1/50

[D] 1/100

[E] None of the above.

15. We wish to break a group of n students into project-teams. Each team must have either 4 or 6 students.

[A] IF $n \geq 4$, THEN it can be done.

[B] IF $n \geq 6$, THEN it can be done.

[C] IF $n \geq 10$, THEN it can be done.

[D] IF $n \geq 4$ and n is even, THEN it can be done.

[E] None of the above.

$P(n) \to P(n+4)$

$\boxed{P(4)} \quad \boxed{P(6)} \quad \boxed{P(8)} \quad \boxed{P(10)} \quad \cdots$

Can't do odd as $4x + 6y$ is always even

$\therefore$ can do any even ≥ 4.

16. What are the first four terms A_0, A_1, A_2, A_3 in the the recurrence $A_n = \begin{cases} 1 & n = 0; \\ 2A_{n-1} + 1 & n \geq 1. \end{cases}$

$\boxed{\text{A}}$ $1, 2, 3, 4.$

$\boxed{\text{B}}$ $1, 2, 4, 8.$

$\boxed{\text{C}}$ $1, 3, 6, 12.$

$\boxed{\text{D}}$ $1, 3, 7, 15.$

$\boxed{\text{E}}$ None of the above.

1 3 7 15 . . . .

17. For $n \geq 0$, what is a formula for A_n, where A_n satisfies the recurrence $A_n = \begin{cases} 1 & n = 0; \\ 2A_{n-1} + 1 & n \geq 1. \end{cases}$

$\boxed{\text{A}}$ $A_n = 1 + 2n$ for $n \geq 0$.

$\boxed{\text{B}}$ $A_n = 1 + n + n^2$ for $n \geq 0$.

$\boxed{\text{C}}$ $A_n = 1 + \frac{1}{3}(5n + n^3)$ for $n \geq 0$.

$\boxed{\text{D}}$ $A_n = 2^{n+1} - 1$ for $n \geq 0$.

$\boxed{\text{E}}$ None of the above.

$A_n = 2^{n+1} - 1$

Proof by induction

$A_0 = 2^1 - 1 = 1$ ✓

Assume $P(n)$: $A_n = 2^{n+1} - 1$

Prove $P(n+1)$: $A_{n+1} = 2^{n+2} - 1$

$A_{n+1} = 2A_n + 1$
$= 2 \cdot 2^{n+1} - 2 + 1$
$= 2^{n+2} - 1$ ✓.

18. String x is a palindrome, that is $x = x^{\text{R}}$ where x^{R} is the reversal of x. Which statement about x is **false**?

$\boxed{\text{A}}$ x could be the string 1001.

$\boxed{\text{B}}$ The reversal of x must be a palindrome, that is x^{R} is a palindrome.

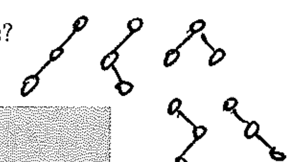
$\boxed{\text{C}}$ The concatenation of x with itself is a palindrome, that is $x \bullet x$ is a palindrome.

$\boxed{\text{D}}$ x must have even length.

$\boxed{\text{E}}$ The concatenation of x with its reversal is a palindrome, that is $x \bullet x^{\text{R}}$ is a palindrome.

010 is a palindrome.
∴ D is false.

19. Rooted binary trees (RBTs) are recursively defined below. How many RBTs have 3 vertices?



5.

$\boxed{\text{A}}$ 2

$\boxed{\text{B}}$ 3

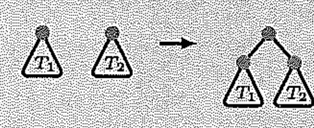
$\boxed{\text{C}}$ 4

$\boxed{\text{D}}$ 5

$\boxed{\text{E}}$ 6

Recursive Definition of RBT

① The empty tree ε is an RBT.
② If T_1, T_2 are disjoint RBTs with roots r_1 and r_2, then linking r_1 and r_2 to a *new* root r gives a new RBT with root r.
③ Nothing else is an RBT.



20. A rooted binary tree (RBT) has 8 vertices. How many links (edges) does the RBT have?

$\boxed{\text{A}}$ There is not enough information to determine the number of links.

$\boxed{\text{B}}$ 5

$\boxed{\text{C}}$ 6

$\boxed{\text{D}}$ 7

$\boxed{\text{E}}$ 8

links = # vertices − 1
↑
8
links = 8 − 1 = 7.