QUIZ 2: 60 Minutes

Last Name: Solutions

First Name: _____

RIN: _____

Section: _____

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

Total
200

1. How many students would guarantee two students with the same first initial. (First initials are $A, B, \ldots, Z$.)

A 24.

B 25.

C 26.

D 27.

E None of the above.

D

Pigeonholes 26

Pigeons 27 ⟶ two in same pigeonhole.

2. How many numbers in $\{1, 2, \ldots, 1000\}$ are divisible by 2 or 5.

A 200.

B 500.

C 600.

D 700.

E None of the above.

C

NO WORK, NO CREDIT

A_2 = div by 2 = $\lfloor \frac{1000}{2} \rfloor = 500$

A_5 = div by 5 = $\lfloor \frac{1000}{5} \rfloor = 200$

$A_{2,5}$ = div by 2,5 = $\lfloor \frac{1000}{\text{lcm}(2,5)} \rfloor = \lfloor \frac{1000}{10} \rfloor = 100.$

Inclusion Ex: $500 + 200 - 100 = 600.$

3. In how many ways can you pick a cooking team of 4 students from 7 student-chefs?

A 4×7.

B $7!/4!$.

C 7^4.

D $7!/(4! \times 3!)$.

E None of the above.

D

$\binom{7}{4} = \frac{7!}{4!\,3!}$

4. The only available majors at FOCS-University are CS and BIO. There are 70 students in total. There are 50 CS majors and 50 BIO majors. How many dual CS-BIO majors are there?

A 10.

B 20.

C 30.

D 40.

E None of the above.

C

NO WORK NO CREDIT

$|A| = 50 \qquad |B| = 50$

$|A \cup B| = 70 = |A| + |B| - |A \cap B|$

$= 50 + 50 - |A \cap B|$

$\longrightarrow |A \cap B| = 100 - 70 = 30.$

5. The sum of the probabilities for all possible outcomes is always:

A 0

B 1/4.

C 1/2.

D 1.

E None of the above.

D

6. Randomly flip three independent fair coins. What are the chances of more H (heads)?

A 0.

B 2/8.

C 4/8.

D 6/8.

E 1.

C

NO WORK NO CREDIT

More heads → 2 H or 3 H

HHH HHT HTH HTT THH THT TTH TTT

1/8 1/8 1/8 1/8 1/8 1/8 1/8 1/8

ADD = 1/8 × 4 = 1/2 = 4/8

7. Randomly flip three independent fair coins. What are the chances of more H if the first flip is H?

A 0.

B 2/8.

C 4/8.

D 6/8.

E 1.

D

NO WORK NO CREDIT

$P[\text{more H} \mid H] = \frac{P[\text{more H} \wedge H]}{P[H]} = \frac{P[\{HHH, HHT, HTH\}]}{P[\{HHH, HHT, HTH, HTT\}]}$

$= \frac{3/8}{4/8} = \frac{3}{4} = \frac{6}{8}$

8. Randomly flip three independent fair coins. What are the chances of more H if the first two flips are H?

A 0.

B 2/8.

C 4/8.

D 6/8.

E 1.

E

First 2 H → More heads

∴ Prob = 1.

9. Random variable **X** is a value in $\{1, 2, 3, 4\}$ with probabilities $\{x, 2x, 3x, 4x\}$ respectively. What is x?

A 0.1.

B 0.2.

C 0.3.

D 0.4.

E It cannot be determined or none of the above.

A

$x + 2x + 3x + 4x = 1 \rightarrow 10x = 1 \rightarrow x = \frac{1}{10} = 0.1$

10. For the random variable **X** in the previous problem, what is $\mathbb{E}[\mathbf{X}]$

A 1.

B 2.

C 3.

D 4.

E It cannot be determined or none of the above.

C

NO WORK NO CREDIT

$E[X] = 1 \cdot x + 2 \cdot 2x + 3 \cdot 3x + 4 \cdot 4x$

$= x[1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4]$

$= \frac{1}{10} \cdot [30] = 3.$

11. What is the probability to get 4 or more heads in 5 independent flips of a biased coin with probability 1/3 of heads.

$$\binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) + \binom{5}{5}\left(\frac{1}{3}\right)^5$$

$$= \frac{1}{3^5}\left[\binom{5}{4}2 + \binom{5}{5}\cdot 1\right] = \frac{1}{3^5}\left[5\cdot 2 + 1\right] = \frac{11}{3^5}$$

[A] $10/3^5$.

[B] $11/3^5$.

[C] $12/3^5$.

[D] $13/3^5$.

[E] None of the above.

B

NO WORK NO CREDIT

12. You flip a fair coin 10 times. What is the expected number of heads? ↙ Binomial.

[A] 3.

[B] 4.

[C] 5.

[D] 6.

[E] None of the above.

C

$n=10$ $\quad p=\frac{1}{2}$ $\quad np = 5 = E[\#\text{ successes}]$

13. Boys are 4 times as likely as girls. You have kids till you get a girl. What is the expected number of kids you will have?

[A] 2.

[B] 3.

[C] 4.

[D] 5.

[E] None of the above.

D

$4p$ = Prob Boy, p = Prob Girl. add: $5p = 1 \rightarrow p = \frac{1}{5}$

$$E[\text{wait}] = \frac{1}{p} = 5.$$

14. Random variables $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ have expected values $\mu_1 = 1, \mu_2 = 2, \mu_3 = 3$. What is $\mathbb{E}[2\mathbf{X}_1 + \mathbf{X}_2 + 3\mathbf{X}_3]$?

[A] 11.

[B] 12.

[C] 13.

[D] 14.

[E] Can't be determined or none of the above.

C

Linearity of expectation

$$E[2X_1 + X_2 + 3X_3] = 2\mu_1 + \mu_2 + 3\mu_3$$
$$= 2\cdot 1 + 2 + 3\cdot 3$$
$$= 13$$

15. 60% of students are men and 40% are women. Men have average hair length 5in. Women have average hair length 10in. What is the average hair length for all students?

[A] 6in.

[B] 7in.

[C] 8in.

[D] 9in.

[E] None of the above.

B

NO WORK NO CREDIT

$$E[\text{length}] = E[\text{length}|\text{man}]\,P[\text{man}] + E[\text{length}|\text{woman}]\,P[\text{woman}]$$
$$5 \qquad 0.6 \qquad 10 \qquad 0.4$$
$$= 5\times 0.6 + 10\times 0.4 = 3 + 4 = 7\text{in}$$

16. A box has 5 fair and 5 two-headed coins Pick a random coin and flip 5 times. What are the chances of exactly 2 heads?

[A] 2/5
[B] 1/32
[C] 5/32
[D] 10/32
[E] None of the above.

C

NO WORK NO CREDIT

$P[2H] = P[2H \mid \text{fair}]P[\text{fair}] + P[2H \mid \text{biased}]P[\text{biased}]$

$\binom{5}{2}\frac{1}{2^5} \quad \frac{1}{2} \quad 0 \quad \frac{1}{2}$

$= \frac{1}{2^6}\binom{5}{2} = \frac{10}{2^6} = \frac{5}{25} = \frac{5}{32}.$

17. A box has 5 fair and 5 two-headed coins Pick a random coin and flip 10 times. What is the expected number of heads?

[A] 5.
[B] 6.
[C] 7.
[D] 8.
[E] None of the above.

E

$E[\text{heads}] = E[\text{heads} \mid \text{fair}]P[\text{fair}] + E[\text{heads} \mid \text{biased}]P[\text{biased}]$

$= 5 \quad \frac{1}{2} \quad 10 \quad \frac{1}{2}$

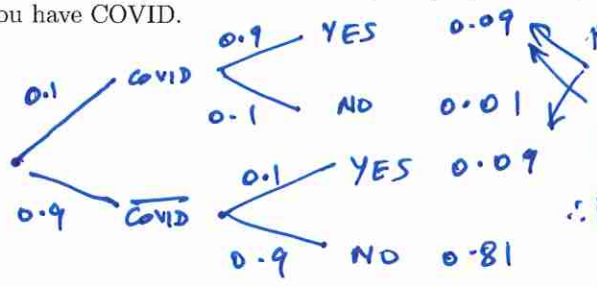
$= 5 + \frac{5}{2} = \frac{15}{2} = 7.5$

18. A test for COVID gives the right answer 90% of the time and 10% of people have COVID. You tested positive. What are the chances you have COVID.

[A] 9%.
[B] 10%.
[C] 50%.
[D] 90%.
[E] None of the above.

C.

NO WORK NO CREDIT



$P[\text{yes}] = 0.18$

$P[\text{COVID} \wedge \text{YES}] = 0.09$

$\therefore P[\text{COVID} \mid \text{YES}] = \frac{0.09}{0.18} = \frac{1}{2}.$

19. A test for COVID gives the right answer 90% of the time and 10% of people have COVID. You test 100 people. What is the expected number people that have a positive test.

[A] 9.
[B] 10.
[C] 18.
[D] 90.
[E] None of the above.

C

NO WORK NO CREDIT

100 people $X_1 \; X_2 \; \ldots \; X_{100}$ ← 100 indicators of whether test positive.

$X = X_1 + X_2 + \ldots + X_{100}$

$E[X] = E[X_1] + \ldots + E[X_{100}] = 100 \times E[X_1] \longrightarrow 100 \times 0.18 = 18.$

$E[X_1] = E[X_1 \mid \text{covid}]P[\text{covid}] + E[X_1 \mid \overline{\text{covid}}]P[\overline{\text{covid}}] = 0.18$

$0.9 \quad 0.1 \quad 0.1 \quad 0.9$

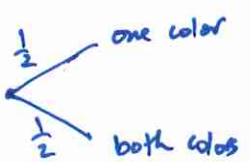
20. Starburst comes in two-packs and there are two equally likely colors. You buy two-packs until you have at least one starbust of each color. What is the expected number of two-packs you buy?

[A] 4/3.
[B] 5/3.
[C] 6/3.
[D] 7/3.
[E] None of the above.

B

NO WORK NO CREDIT

The first pack has both colors or one color: $\frac{1}{2}$ $\frac{1}{2}$



$E[\text{wait}] = E[\text{wait} \mid \text{one}]P[\text{one}] + E[\text{wait} \mid \text{both}]P[\text{both}].$

waiting for 1 $\quad \frac{1}{2} \quad 1 \quad \frac{1}{2}$

Succ prob $\frac{3}{4}$

$= \left(1 + \frac{4}{3}\right) \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{5}{3}$