

# Pricing the Quality Option for the Bond Futures Contract in a Multifactor Vasicek Framework

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## Abstract

We present a study of the pricing of the bond futures contract. More specifically, we study the pricing of the bond futures *quality* option within a  $k$ -factor Vasicek interest rate model. We demonstrate how existing approaches are inadequate for applications that one might be interested in and demonstrate a new technique that not only prices the contracts more accurately, taking the quality option into account, but does so in a computationally efficient way. As such, it is well suited for use in calibration of such  $k$ -factor Vasicek interest rate models to observed futures price data.

## 1 Introduction:

The US Treasury bond (and note) futures are some of the most followed and heavily traded futures contracts in the world. Even though pricing most futures contracts has been a straightforward problem for most contracts, and can be obtained by the cost-of-carry method, this has not been the case for the Treasury bond futures. Such contracts possess *delivery options*, that give the short the right to choose the bond to deliver from among a basket of bonds, in addition to choosing the time

at which to deliver that bond. Such choices give the futures contracts both european and american style options features. Even for very simple models of the underlying interest rate, no closed form solution has been developed. Needless to say that an accurate formula will be very valuable for Treasury futures traders.

The majority of methods developed to tackle the delivery options (e.g. [3], [2], [6], and [4]) are based on very simple one-factor interest rate models, and are all approximations. In this paper we propose a new accurate method for pricing the Treasury bond futures. We use a general  $k$ -factor interest rate model (the  $k$ -factor Vasicek model). A  $k$ -factor interest rate model is a way to model the interest rate across the different maturities using  $k$  random components ([5]). It is modeled by  $k$  stochastic differential equations (akin to stocks being modeled by a Wiener process, except that the interest rate model is mean reverting since interest rates typically move in a certain range). A one-factor model is not adequate because it results in only one-degree-of-freedom for movements of the entire yield curve (interest versus maturity), which is typically a vertical shift in the daily evolution of the yield curve. Three or more factors is usually more realistic, because they can account for level shifts, linear shifts, and bends in the daily yield curve dynamics, which are typically the types of moves empirically observed.

## 2 Delivery Options

There are the following four types of delivery options for the Treasury bond futures ([1]):

1. The quality option: The CBOT (Chicago Board of Trade Exchange) exchange prespecifies a basket of bonds of different maturities and coupon values as acceptable deliverables at the futures contract maturity. The short has the option of choosing which bond within this basket to deliver. Naturally, the short will choose to deliver the cheapest bond in the basket.
2. The switching option: The last trading day is the 8th business day before the end of the contract month, whereas the last delivery day is the last business day in the month. While in the intervening days between these two dates, the futures price is no longer free to vary with market forces, the deliverable bonds do vary in price, and allow the short to “switch” the deliverable from one bond to another during this interval.
3. The timing option: According to the exchange, delivery can take place any day in the delivery month, and the short gets to choose that day.

4. The wild card option: The deadline for the short to give notice of delivery is the day previous to delivery on 8 pm. However, the futures market closes at 2 pm, and after that time the settlement price is fixed. If after 2 pm the price of the cheapest to deliver bond drops significantly, it is advantageous for the short to buy the bond and give notice of delivery rather than wait until the next trading day when the futures price will be lower because of the drop in cash prices (this can be considered as an arbitrage opportunity).

Delivery options vary in importance. The quality option is by far the most important of the options. Following that in the order of importance are the switching option, the wild card option and the timing option. The timing option is almost worthless, for, it can be shown that in almost all cases, it is advantageous to deliver on the last business day of the month. For this reason, in this paper we consider only the quality option.

### 3 Vasicek $k$ -factor Models

Vasicek  $k$ -factor models assume the variable of interest to be driven by  $k$  “hidden” factors or levels, each of which exhibits mean reverting behavior that is driven by a Wiener process. These hidden factors essentially form a generalized  $k$ -dimensional Orstein-Uhlenbeck process. Thus, we assume the interest rate  $r(t)$  to follow the following risk neutral dynamics

$$r(t) = \sum_{i=1}^k x_i(t) \tag{1}$$

where the  $x_i(t)$  are the hidden factors that form the Orstein-Uhlenbeck process and are governed by the dynamics

$$dx_i(t) = a_i(b_i - x_i(t))dt + \sigma_i dW_i(t), \quad i = 1, \dots, k \tag{2}$$

where each  $b_i$  represents the level to which each  $x_i$  reverts, with time constants  $1/a_i$ .  $\sigma_i$  represents the volatility of each factor and the  $dW_i(t)$ 's represent standard white noise / Wiener processes that are not necessarily uncorrelated, i.e.,

$$E [dW_i(t)dW_j(t)] = \rho_{ij}dt \tag{3}$$

where  $\rho$  is a correlation matrix with  $\rho_{ii} = 1$ . The advantage of using a multi-factor ( $k > 1$ ) rather than a single-factor model is that the multi-factor model allows for the interest rate to adjust to (possibly correlated) shocks represented by the Wiener's at different rates that are related to the time constants of the equations. This becomes of

practical importance for trading purposes. If one is interested in placing trades on a monthly basis, then one might be satisfied with a model that captures the behavior on monthly and yearly scales, hence a 2-factor model would suffice. However, for daily trading, one would also need to capture the daily dynamics and thus an extra factor might be in order, or for intra-day trading one might need four or maybe five factors.

How does one fix the parameters in the model? This is the process known as calibration, where one observes the market prices of various futures contracts and obtains model parameters that attempt to reproduce the market prices. One could then determine, for example, which contracts are overvalued and which ones are undervalued, and trade accordingly. Effective calibration of such models will be discussed elsewhere, however, we wish to emphasize that a prerequisite to being able to efficiently calibrate is that one must be able to efficiently price the contract. The two requirements are that the pricing must be accurate and fast. Accuracy is necessary to obtain sound trading decisions (which depend on an accurate estimate of the fair value of the contract). Calibration usually requires a search through the parameter space that is terminated when a set of parameters that adequately reflects the market prices is found. This usually means that the contracts will have to be priced for many sets of parameters, hence the speed requirement. The frequently used existing techniques lack one or the other of these requirements, but the technique that we propose satisfies both requirements.

## 4 Pricing the Bond Futures Contract

As mentioned in Section 2, we propose a method based on a  $k$ -factor Vasicek model to price the futures contract. The method take into account the quality option.

A frequently used pricing method is to price the option as if the bond that will be delivered is the bond that would be cheapest to deliver today. This approximation yields a closed form expression in this Vasicek  $k$ -factor formulation. It is useful as a first approximation because it is fast to compute, however it is based on the assumption that the cheapest to deliver today will be the cheapest to deliver at the time of delivery. This method ignores any independence among the bonds in the basket, and, thus, we expect it to work well if there is a large gap in value between the cheapest to deliver today and the remaining bonds in the basket, or, if the delivery horizon is very close. On the other hand, this estimate can be quite inaccurate if the time to delivery is large or if the bonds in the basket are priced approximately equally, despite being somewhat independent. Thus, though this method is

fast (as a closed form expression for the futures price can be derived in this Vasicek framework), it can be quite inaccurate, which fails one of our requirements in order to be able to calibrate effectively.

The other frequently used method is to simulate risk neutral interest rate paths in a Monte Carlo fashion up to the time to delivery. At this point, the cheapest to deliver can be determined and the value of the contract resulting from this particular path can be discounted along the path to the present time. Taking an expectation over possible paths, one obtains the risk neutral valuation of the contract (assuming, as we have, that the risk neutral dynamics are given by a  $k$ -factor Vasicek model). This estimate is accurate, provided that enough paths are taken. Unfortunately, the number of paths needed for a reasonably accurate price is quite significant (about  $10^6$ ). Although this could be manageable if we are pricing a single contract given a set of parameters, it is completely out of the question for the calibration process, which is essentially an optimization process that entails thousands of evaluations of the contract value (each evaluation being performed using a separate Monte Carlo simulation).

We have implemented an approach that combines aspects of both these methods. Our approach incorporates more fully the independence of the bonds in the basket (unlike the pricing of the current cheapest to deliver method), yet, we are able to extract a closed form for the price, making the computation fast as well (unlike the Monte Carlo approach). As such, it is an approach that satisfies both the requirements for calibration, and is thus superior to the two frequently used existing methods.

Unfortunately, as can be expected, the formulas for the pricing of the minimum, the Monte Carlo approach and our new hybrid method would take pages to write down let alone derive (for this  $k$ -factor Vasicek framework), and thus, we postpone these details to a future presentation. We present in the next section the results of the simulations that have been performed for a basket of each bonds of approximately 10 years till maturity.

## 5 Simulation Results

We have created a basket of 10 approximately ten year bonds with coupons six months apart that are very similar to the deliverable basket actually encountered for ten year bond futures. We then price these bonds using all three methods discussed above (for our proposed method we considered a *three*-factor Vasicek model). For the Monte Carlo simulation, 5 million paths were used. Figure 1 illustrates the difference between the Monte Carlo and our new approach. We take the Monte Carlo as being the ground truth, in which case we see that

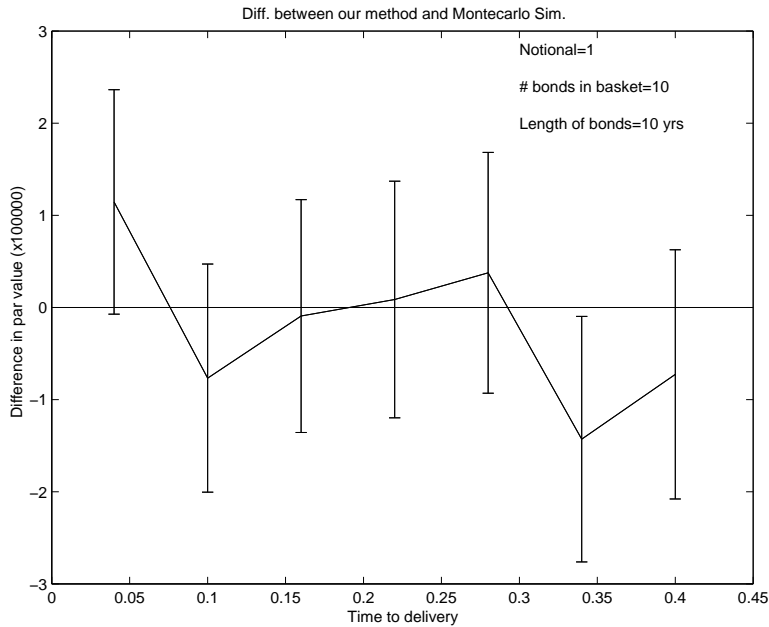


Figure 1: Comparison between our hybrid technique and the Monte Carlo simulation. The error bars reflect the one sigma statistical fluctuations in the Monte Carlo estimate.

the difference is of order 0.1 basis point (1 bp=0.01, assuming the notional or face value of the bond is 100), which is also the order of the statistical fluctuations in the Monte Carlo estimate with this number of paths. The accuracy of our hybrid method appears to be stable even as one increases the time to delivery. Figure 2 shows a comparison between the minimum to deliver method and our method (which we have seen is essentially equal to the ground truth Monte Carlo). As was already mentioned, for short times to delivery, the pricing of the minimum gives reasonably accurate results, but as the time to delivery increases, the pricing of the minimum becomes increasingly less accurate, with errors greater than 5 bp as time to delivery approaches half a year. This is an order of magnitude greater than the error of the hybrid method.

## 6 Conclusions

In this presentation, the intention was to demonstrate that fast and accurate pricing for the bond futures contract can be obtained using a Vasicek  $k$ -factor model, hence making it feasible to calibrate multi-

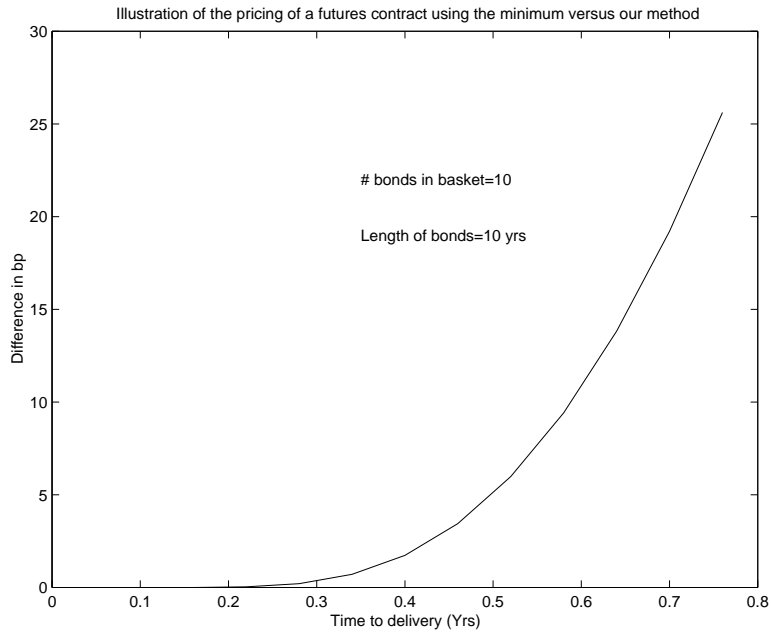


Figure 2: Comparison between our hybrid method and the pricing of the minimum today approach.

factor Vasicek models to futures data. The simulations have pointed out some of the drawbacks that the existing methods suffer from. The pricing of the minimum can be inaccurate, and the Monte Carlo with 5 million paths is still only accurate to about one tenth of a basis point, though it is an extremely computationally expensive procedure for calibration. The new hybrid method, on the other hand, is both fast and accurate.

## References

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