Learning Martingale Measures to Price Options

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Abstract

We provide a framework for learning risk-neutral measures (Martingale measures) for pricing options. In a simple geometric Brownian motion model, the price volatility, fixed interest rate and a no-arbitrage condition suffice to determine a unique risk-neutral measure. On the other hand, in our framework, we relax some of these assumptions to obtain a *class* of allowable risk-neutral measures. We then propose a framework for learning the appropriate risk-neural measure. Since the risk-neutral measure prices all options simultaneously, we can use all the option contracts on a particular stock for learning. We demonstrate the performance of these models on historical data. In particular, we show that both learning without a no-arbitrage condition and a no-arbitrage condition without learning are worse than our framework; however the combination of learning with a no-arbitrage condition has the best result. These results indicate the potential to learn Martingale measures with a no-arbitrage condition providing just the right constraint. We also compare our approach to standard Binomial models with volatility estimates (historical volatility and GARCH volatility predictors). Finally, we illustrate the power of such a framework by developing a real time trading system based upon these pricing methods.

1 Introduction

In 1973, Black and Scholes published their pioneering paper [3] which introduced the first option pricing formula and also developed a general framework for derivative pricing. Since then, derivative pricing has become a popular research topic. A modern, popular approach to pricing has been though the Martingale measure (see, for example, [9]). The origin of the fundamental theorems on the Martingale measure can be traced to Cox and Ross' paper [4] describing the method of *risk neutral valuation*. The Martingale measure was developed into a more mature pricing technique in [1, 6, 7, 8]. Other related topics can be found in [9, 10]. Often the Martingale measure is not unique, and we develop a framework for *learning* the Martingale measure. Within this framework, the same Martingale measure is used to price *all* derivatives of the same underlying stock. This means that data on *all* derivatives of the same underlying stock can be used for *learning* within this framework. We only use the American call and put options in our experiments. We use these pricing algorithms to develop a trading strategy and measure the performance of the pricing by the profit of the trading. The outline of this paper is as follows: first, we introduce the two period economy and some notation; we continue by introducing the definition of arbitrage and the fundamental theorems of risk neutral pricing, which are the backbone of our framework. Next, we present two models for option pricing. The first model is the binomial model, introduced by Cox, Ross and Rubinstein [5], and further information can be found in [9, 11, 12]. We also consider the trinomial model which is more complicated, and more flexible than the binomial, [2]. In both models, we discuss what needs to be learned and how to use Martingale measures to compute option prices. For background on option pricing and other financial topics, we suggest [2, 11, 12]. Finally, we present some results on the performance of our approach as compared with other algorithms.

2 Two Period Economy

Before introducing the Martingale measure, we need set up the notation to describe the economy. Suppose that there are N instruments at time t, whose price is given by $S^i(t)$, $i = 1, \ldots, N$. For the moment, let's only consider a two period economy, $t = 0$ and $t = T$. The instrument price $S^i(0)$, after a period of time T, has K possible states. $S^i_j(T)$, where j=1, …, K indexes each possible state. The probability of state j occurring is P_j, where j=1, …, K and $\sum P_j = 1$. We can represent $S^i_j(T)$ and P_j in vector notation,

$$\mathbf{S}_j(T) = \begin{bmatrix} S^1_j(T) \\ S^2_j(T) \\ \vdots \\ S^N_j(T) \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_K \end{bmatrix}.$$

We define the payoff matrix as

$$\mathbf{Z}(T) = [\mathbf{S}_1(T), \mathbf{S}_2(T), \cdots, \mathbf{S}_K(T)] = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NK} \end{bmatrix},$$

where Z_{ij} indicates the price of $S^i(T)$ in one possible state j of the economy. In other words, the instrument prices $\mathbf{S}(0)$ at time 0 have probability P_j to be $\mathbf{S}_j(T)$ at time T.

3 No Arbitrage

We now define arbitrage. Intuitively, arbitrage is the possibility to make money out of nothing. The formal definitions are follows (see, for example, [9]). A portfolio $\mathbf{\Theta}$ is a column vector of N components which denotes how many units of each instrument is held, and we use the notation $(\cdot)^T$ for the transpose.

Definition 3.1 (Type I Arbitrage): An *arbitrage opportunity* of type I exists if and only if there exist a portfolio $\mathbf{\Theta}$ such that[1]

$$\mathbf{\Theta}^T\mathbf{S}(0) \le 0 \quad \text{and} \quad \mathbf{\Theta}^T\boldsymbol{Z}(T) \ge 0.$$

Definition 3.2 (Type II Arbitrage): An *arbitrage opportunity* of type II exists if and only if there exist a portfolio $\mathbf{\Theta}$ such that

$$\mathbf{\Theta}^T\mathbf{S}(0) < 0 \quad \text{and} \quad \mathbf{\Theta}^T\boldsymbol{Z}(T) \geqq 0.$$

In words, a type I arbitrage opportunity has a negative or zero investment today and a nonnegative return in the future with at least one possible positive return state; a type II arbitrage opportunity has a negative investment today and a nonnegative return in the future. Therefore, if there is an arbitrage in the economy, there is no risk for anyone investing in this arbitrage portfolio, and hence every individual will want to consume an infinite amount of such portfolios, creating disequilibrium. Accordingly, it is natural to disallow such arbitrage opportunities.

4 The Risk-Neutral/Martingale Measure

Based on the following two fundamental theorems (see, for example, [9]), we can determine a Martingale measure for pricing.

Theorem 4.1 (Positive Supporting Price): The following statements are equivalent.

[1]For vectors, the notation $\mathbf{V} \ge 0$ indicates each component ≥ 0 and at least one component > 0. The notation $\mathbf{V} \geqq 0$ indicates each component ≥ 0, and there is a possibility that every component $= 0$.

1. There do not exist arbitrage opportunities of type I or type II.
2. There exists a column vector $\boldsymbol{\psi} > 0$ such that

$$\mathbf{S}(0) = \mathbf{Z}(T)\boldsymbol{\psi}. \tag{1}$$

Theorem 4.2 (Equivalent Martingale Measure): There do not exist arbitrage opportunities of type I or II if and only if there exists a probability vector $\tilde{\mathbf{P}}$, called an *equivalent martingale measure* such that

$$\frac{S^i(0)}{S^1(0)} = E_{\tilde{\mathbf{P}}}\left[\frac{S^i(T)}{S^1(T)}\right]. \tag{2}$$

In other words, $S^i(T)/S^1(T)$ is a martingale under the measure $\tilde{\mathbf{P}}$. Sometimes, the measure $\tilde{\mathbf{P}}$ is also called the *risk-neural measure* or the *risk-adjusted probabilities*. From equation (1), after some rearrangement, we can obtain $\tilde{P}_i$ in term of ψ_i,

$$\tilde{P}_i = \frac{Z_{1i}}{S^1(0)}\psi_i \quad \text{and} \quad \sum \tilde{P}_i = 1.$$

5 Option Pricing

We will always assume the existence of a *risk free* asset or *bond*, B, which has the property that the price of the bond is $B(0)$ at time 0, and it has the same value in all states at time T, $B_1(T) = B_j(T)$, where $j = 1, \ldots, K$; simplify the notation $B_j(T)$ to $B(T)$, and define the *risk free discount factor*, $D(T) = B(0)/B(T)$. If we use the bond B as the instrument S^1, then the equation (2) becomes

$$S^i(0) = D(T) \times E_{\tilde{\mathbf{P}}}\left[S^i(T)\right], \qquad i = 1, 2, \ldots, K\ , \tag{3}$$

which means that the current prices are the present value of the expected future prices, where the expectation is with respect to the *risk-neural probability measure*.

5.1 Binomial (2-State) Model

The simplest model of a geometric Brownian motion is the binomial model, [12, 11]. In this model, during each time step, the price of instrument can only move up or down (Figure 1.(a)). Let's consider an economy that has three instruments, one *stock* S, one *bond* B, and one *derivative* C. Based on the prices of *stock* S and *bond* B, and using the equation (3), we can discover that

$$S(0) = D(T) \times \left(\tilde{P}_1 S_1(T) + (1 - \tilde{P}_1) S_2(T)\right). \tag{4}$$

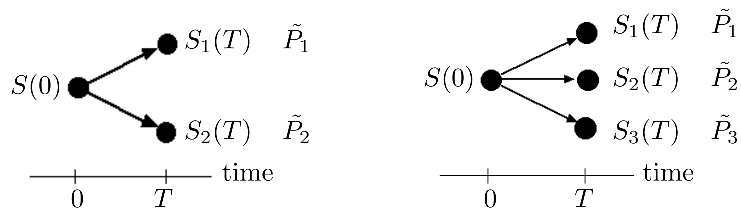
Therefore, if we know the values[2], $S_1(T)$, $S_2(T)$, and $B(T)$, and use the current price for $S(0)$ and $B(0)$, we can compute the unique *risk-neural probability* from equation (4)

$$\tilde{P}_1 = \frac{\left(\frac{S(0)}{D(T)}\right) - S_2(T)}{S_1(T) - S_2(T)}.$$

In this economy, the Martingale measure is unique and all derivatives of *stock* S can be priced with *risk-neural probability* $\tilde{\mathbf{P}}$ and equation (4). For instance, if C is a derivative whose values at time T are known (eg. a call option), then

$$C(0) = D(T) \times \left(\tilde{P}_1 C_1(T) + (1 - \tilde{P}_1) C_2(T)\right).$$

[2] There are many techniques to determine appropriate values for $S_1(T)$ and $S_2(T)$, such as historical volatility and GARCH volatility predictors.



(a) Binomial (2-State) economy (b) Trinomial (3-State) economy

Figure 1: The Dynamics of Economy

5.2 Trinomial (3-State) Model

Now consider the trinomial model (Figure 1.(b)) in which (as we will see) the Martingale measure is not unique. The price $S(0)$ can change to 3 possible values at time T. Following the same argument as in the binomial model and applying equation (3) in an economy with the same three instruments, S, B, and C, we obtain

$$S(0) = D(T) \times \left(\tilde{P}_1 S_1(T) + \tilde{P}_2 S_2(T) + \tilde{P}_3 S_3(T)\right), \tag{5}$$

and since $\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 = 1$,

$$\tilde{P}_2 = \frac{\left(\frac{S(0)}{D(T)}\right) - S_3(T)}{S_2(T) - S_3(T)} - \tilde{P}_1 \times \frac{S_1(T) - S_3(T)}{S_2(T) - S_3(T)}, \tag{6}$$

$$\tilde{P}_3 = \frac{\left(\frac{S(0)}{D(T)}\right) - S_2(T)}{S_3(T) - S_2(T)} - \tilde{P}_1 \times \frac{S_1(T) - S_2(T)}{S_3(T) - S_2(T)}. \tag{7}$$

From equations (6), (7) and the facts that $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_3 > 0$, we can only obtain a range for $\tilde{P}_1$: $\tilde{P}_1 \in [min, max]$. Thus, $\tilde{P}_1$ is not uniquely defined. The problem becomes more complicated; on the other hand, the model becomes more flexible, and we can now try to appropriately learn the Martingale measures from more information, to obtain a better pricing for the instruments at time 0. Our work combines learning with the appropriate no-arbitrage constraints (eg. (6),(7)) to arrive at better option pricing.

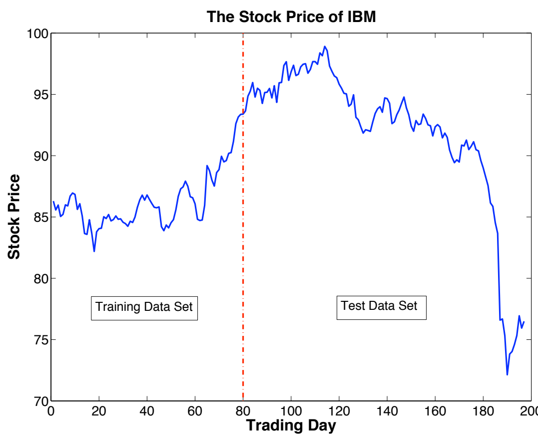
6 Results

We developed a simple trading system to evaluate our framework, which we tested using intraday real market data for IBM (stock and option data) and interest rate data, from July 20, 2004 to April 29, 2005. We used the first 80 days, from July 20, 2004 to November 9, 2004, as the training data set, and used the remaining 118 days as test data (Figure 2.(a)). We compared the trading performance between different algorithms.

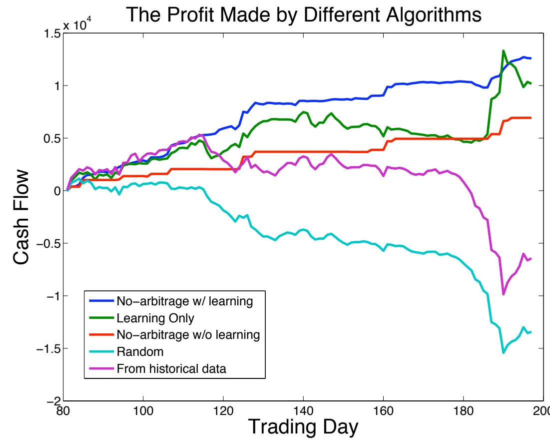
1. **Enforcing No-arbitrage, with learning**: This is our framework which is based on a no-arbitrage condition, and also a learning algorithm to predict the Martingale measure.

2. **Not Enforcing No-arbitrage, with learning**: This approach is only based on the learning algorithm without no-arbitrage constraints.

3. **Enforcing No-arbitrage, no learning**: This approach is to demonstrate that a no-arbitrage constraint alone, without learning the Martingale measures is worse than our framework.



(a) The market price of IBM



(b) Comparison between different pricing algorithms

Figure 2: The trading results of stock IBM using three different algorithms

4. **Not Enforcing No-arbitrage, no learning (random strategy)**: This approach is to develop a benchmark performance using a random strategy.
5. **Not Enforcing No-arbitrage, learning the probabilities of each state of the stock as the risk-neutral probabilities**: This approach is to demonstrate that pricing should be based on risk-netutral probabilities, not the probabilities of each state of the stock.

The results of trading using these approaches are shown in figure (Figure 2.(b)). Our framework clearly has the best performance. Note that the system still makes money even when the market crashes. As we move further from the training window, the performance degrades, though it remains positive. The results of the other algorithms are also reasonable because any random trading strategy will systematically lose the transaction cost on each trade which means that the total profit will drop linearly; the results also show that it is useful to use a no-arbitrage condition because it narrows the range of Martingale measures to obtain a set of plausible prices, rather than pure random.

7 Conclusions

Our results show that the right constraint (no-arbitrage) for option pricing can enable one to potentially learn the Martingale measure. By using a no-arbitrage condition, we can narrow the range of possible Martingale measures for the learning - the no-arbitrage constraint regularizes the learning in the right direction to yield a better learning outcome. Another benefit of our framework is that one can learn the Martingale measure from *all* data on all derivatives of the same underlying instrument simultaneously. The derivatives of the same underlying instrument should have correlations which our framework can utilize to yield better performance. In addition, by using this framework, the learned Martingale Measure allows us to price all derivatives simultaneously, which would significantly improve the efficiency of pricing. Our future work includes using a moving training window to increase the performance of predicting the option prices, as we observe a degradation further from the training window. We are also expanding the framework to include derivatives from various different financial markets such as Futures, Commodities and many others.

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