Mean and Variance Updates in the Gaussian Approximation  (This derives the mean and variance updates given in Equations 2 and 3 in the main paper. It makes use of the Gaussian integrals from Figure 2 in the paper.)

The update to the mean is obtained by computing the expected value for the updated distribution $p_{t+1}$.

$$
\mu_{t+1} = \frac{1}{A\sigma_t} \int_{-\infty}^{\infty} dv \cdot N\left(\frac{v-\mu_t}{\sigma_t}\right) \left[ \Phi\left(\frac{z^+-v}{\sigma_t}\right) - \Phi\left(\frac{z^-+v}{\sigma_t}\right) \right],
$$

$$
= \frac{1}{A} \int_{-\infty}^{\infty} dv (\mu_t + \sigma_tv) \cdot N(v) \left[ \Phi\left(\frac{z^+-\mu_t-\sigma_tv}{\sigma_t}\right) - \Phi\left(\frac{z^-+\mu_t+\sigma_tv}{\sigma_t}\right) \right],
$$

$$
= \mu_t + \frac{\sigma_t}{A} \left[ J\left(z^+ - \mu_t, \rho_t\right) - J\left(z^- - \mu_t, \rho_t\right) \right].
$$

Thus, we find that

$$
\mu_{t+1} = \mu_t + \frac{\sigma_t}{A} B(z^+, z^-),
$$

To compute the update to the variance, we first compute the second moment,

$$
E[V^2] = \frac{1}{A\sigma_t} \int_{-\infty}^{\infty} dv \cdot v^2 \cdot N\left(\frac{v-\mu_t}{\sigma_t}\right) \left[ \Phi\left(\frac{z^+-v}{\sigma_t}\right) - \Phi\left(\frac{z^-+v}{\sigma_t}\right) \right],
$$

$$
= \frac{1}{A} \int_{-\infty}^{\infty} dv (\mu_t + \sigma_tv)^2 \cdot N(v) \left[ \Phi\left(\frac{z^+-\mu_t-\sigma_tv}{\sigma_t}\right) - \Phi\left(\frac{z^-+\mu_t+\sigma_tv}{\sigma_t}\right) \right],
$$

$$
= \frac{1}{A} \left( \mu_t^2 A + 2\mu_t\sigma_t B + \sigma_t^2 \left[ K\left(\frac{z^+-\mu_t}{\sigma_t}, \rho_t\right) - K\left(\frac{z^-+\mu_t}{\sigma_t}, \rho_t\right) \right] \right).
$$

$$
E[V^2] = \mu_t^2 + \sigma_t^2 + 2\mu_t\sigma_t B - \frac{\sigma_t^2 C}{A}.
$$

Since $\sigma_{t+1}^2 = E[V^2] - \mu_{t+1}^2$, we finally arrive at

$$
\sigma_{t+1}^2 = \sigma_t^2 \left( 1 - \frac{AC + B^2}{A^2} \right).
$$

Proof of Monotonicity of State Update  The proof below is for Theorem 2.1 in the main paper:

Lemma 0.1. For all $x < y$, $(\Phi(y) - \Phi(x))(yN(y) - xN(x)) + (N(x) - N(y))^2 > 0$.

Proof. If $x \leq 0$, the claim is obvious, so assume that $x > 0$. We begin with the following inequality,

$$
x(\Phi(y) - \Phi(x)) < \int_x^y ds sN(s) < y(\Phi(y) - \Phi(x)).
$$
Since $\int_x^y ds \, sN(s) = N(x) - N(y)$, we have the following inequality, which establishes the result,
\[ yN(y)(\Phi(y) - \Phi(x)) - xN(x)(\Phi(y) - \Phi(x)) > N(y)(N(x) - N(y)) - N(x)(N(x) - N(y)) \]

\[ \text{Theorem 0.1 (Monotonic state update). } \sigma_{t+1}^2 \leq \sigma_t^2. \]

\textbf{Proof.} It suffices to show that $AC + B^2 > 0$. Define $s^2 = \sigma_t^2 + \sigma_\epsilon^2$, $\Delta^+ = (z^+ - \mu_t)/s$ and $\Delta^- = (z^- - \mu_t)/s$. After some algebraic manipulation, we find that

\[ A = \Phi(\Delta^+) - \Phi(\Delta^-), \]
\[ B = \frac{\sigma_t}{s} (N(\Delta^-) - N(\Delta^+)), \]
\[ C = \frac{\sigma_\epsilon^2}{s^2} (\Delta^+ N(\Delta^+) - \Delta^- N(\Delta^-)). \]

Thus, $AC + B^2$ is given by

\[ \frac{\sigma_t^2}{s^2} \left[ (\Phi(y) - \Phi(x))(yN(y) - xN(x)) + (N(x) - N(y))^2 \right] \]

where $x = \Delta^-$, $y = \Delta^+$. Applying Lemma 0.1 concludes the proof. 

\[ \blacksquare \]