Supplementary Information for "Adapting to a Market Shock: Optimal Sequential Market-Making"

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Mean and Variance Updates in the Gaussian Approximation (This derives the mean and variance updates given in Equations 2 and 3 in the main paper. It makes use of the Gaussian integrals from Figure 2 in the paper.)

The update to the mean is obtained by computing the expected value for the updated distribution p_{t+1},

$$\begin{aligned}
\mu_{t+1} &= \frac{1}{A\sigma_t}\int_{-\infty}^{\infty} dv\; v \cdot N\left(\tfrac{v-\mu_t}{\sigma_t}\right)\left[\Phi\left(\tfrac{z^+-v}{\sigma_\epsilon}\right) - \Phi\left(\tfrac{z^--v}{\sigma_\epsilon}\right)\right], \\
&= \frac{1}{A}\int_{-\infty}^{\infty} dv\; (\mu+\sigma_t v)\cdot N(v)\left[\Phi\left(\tfrac{z^+-\mu_t-\sigma_t v}{\sigma_\epsilon}\right) - \Phi\left(\tfrac{z^--\mu_t-\sigma_t v}{\sigma_\epsilon}\right)\right], \\
&= \mu_t + \frac{\sigma_t}{A}\left[J\left(\tfrac{z^+-\mu_t}{\sigma_\epsilon},\rho_t\right) - J\left(\tfrac{z^--\mu_t}{\sigma_\epsilon},\rho_t\right)\right].
\end{aligned}$$

Thus, we find that

$$\mu_{t+1} = \mu_t + \sigma_t \cdot \frac{B(z^+,z^-)}{A(z^+,z^-)},$$

To compute the update to the variance, we first compute the second moment,

$$\begin{aligned}
E[V^2] &= \frac{1}{A\sigma_t}\int_{-\infty}^{\infty} dv\; v^2 \cdot N\left(\tfrac{v-\mu_t}{\sigma_t}\right)\left[\Phi\left(\tfrac{z^+-v}{\sigma_\epsilon}\right) - \Phi\left(\tfrac{z^--v}{\sigma_\epsilon}\right)\right], \\
&= \frac{1}{A}\int_{-\infty}^{\infty} dv\; (\mu+\sigma_t v)^2\cdot N(v)\left[\Phi\left(\tfrac{z^+-\mu_t-\sigma_t v}{\sigma_\epsilon}\right) - \Phi\left(\tfrac{z^--\mu_t-\sigma_t v}{\sigma_\epsilon}\right)\right], \\
&= \frac{1}{A}\left(\mu_t^2 A + 2\mu_t\sigma_t B + \sigma_t^2\left[K\left(\tfrac{z^+-\mu_t}{\sigma_\epsilon},\rho_t\right) - K\left(\tfrac{z^--\mu_t}{\sigma_\epsilon},\rho_t\right)\right]\right).
\end{aligned}$$

$$E[V^2] = \mu_t^2 + \sigma_t^2 + 2\mu\sigma_t\frac{B}{A} - \sigma_t^2\frac{C}{A}.$$

Since $\sigma_{t+1}^2 = E[V^2] - \mu_{t+1}^2$, we finally arrive at

$$\sigma_{t+1}^2 = \sigma_t^2\left(1 - \frac{AC+B^2}{A^2}\right).$$

Proof of Monotonicity of State Update The proof below is for Theorem 2.1 in the main paper:

Lemma 0.1. *For all $x<y$, $(\Phi(y)-\Phi(x))(yN(y)-xN(x)) + (N(x)-N(y))^2 > 0$.*

Proof. If $x \le 0$, the claim is obvious, so assume that $x>0$. We begin with the following inequality,

$$x(\Phi(y)-\Phi(x)) < \int_x^y ds\; sN(s) < y(\Phi(y)-\Phi(x)).$$

Since $\int_x^y ds\, sN(s) = N(x) - N(y)$, we have the following inequality, which establishes the result,

$$yN(y)(\Phi(y) - \Phi(x)) - xN(x)(\Phi(y) - \Phi(x)) > N(y)(N(x) - N(y)) - N(x)(N(x) - N(y))$$

■

Theorem 0.1 (Monotonic state update)**.** $\sigma_{t+1}^2 \leq \sigma_t^2$.

Proof. It suffices to show that $AC + B^2 > 0$. Define $s^2 = \sigma_t^2 + \sigma_\epsilon^2$, $\Delta^+ = (z^+ - \mu_t)/s$ and $\Delta^- = (z^- - \mu_t)/s$. After some algebraic manipulation, we find that

$$\begin{aligned} A &= \Phi(\Delta^+) - \Phi(\Delta^-), \\ B &= \frac{\sigma_t}{s}(N(\Delta^-) - N(\Delta^+)), \\ C &= \frac{\sigma_t^2}{s^2}(\Delta^+ N(\Delta^+) - \Delta^- N(\Delta^-)). \end{aligned}$$

Thus, $AC + B^2$ is given by

$$\frac{\sigma_t^2}{s^2}\left[(\Phi(y) - \Phi(x))(yN(y) - xN(x)) + (N(x) - N(y))^2\right]$$

where $x = \Delta^-,\ y = \Delta^+$. Applying Lemma 0.1 concludes the proof. ■