Pricing the American Put Using A New Class of Tight Lower Bounds

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Abstract: *We present new families of lower bounds for the price of the American put option on a dividend paying stock when the stock follows a log normal process and the option can be exercised continuously to a finite horizon T. By put call parity, these bounds can be easily converted to bounds on the price of the Ameican call option on a dividend paying stock. By numerically optimizing these bounds, we obtain tighter bounds on the option price. Our methodolgy simultaneously furnishes us with an (exponential) exercise strategy. We provide an extensive experimental computation, comparing with convergent binomial tree pricing methods. Our bounds deliver an accuracy comparable to a 2000 step binomial tree, with a computational cost comparable to a 400 step binomial tree.*

Keywords: *Put, Call, Optimize, Lower Bound, Exponential Exercise Strategy.*

1 INTRODUCTION

The American put option has been a nemesis in terms of real time pricing and exercise. The main difficulty is dealing with the free boundary problem, and as of yet, no analytic solution has been obtained when the time horizon for exercise is finite. Considerable effort has been devoted to approximating this price. In general, the two key considerations to obtaining accurate prices are to exploit the continuous exercise feature, and to determine the values at which to exercise. With the recent advances in computational technology, a number of approaches based on simulation have also yielded significant progress. The key consideration then becomes the speed accuracy trade off – at the expense of more computing cycles, one can always get more accurate prices. A brief survey of the American option's colorful history follows, and more information can be found in the references.

Johnson, [25], made the first attempt to obtain an analytical approximation to the put option price by fitting a parametric form to a data set of known prices. Similar to this approach, are the ones by [3, 27, 32]. In [22] an analytic approximation is developed by approximating the continuous exercise feature with two exercise opportunities, at $T/2$ and T. This approach was extended in [12] to allow the second exercise time to be chosen optimally. Today, such approaches to developing data-driven formulas has gotten quite sophisticated, especially with the use of powerful genetic programming techniques [26]. A drawback of data driven methods is that the data sets are often chosen arbitrarily, and there is no way to guarantee the performance on new data. As a result, approximating the price using tight lower and upper bounds becomes an attractive choice, (see for example [15, 29, 30, 38]). Upper and lower bounds based on the American capped call option (similar to the 0-point bound we will discuss later) [9] were used in combination with a data driven approach to approximate the price in [10]. As computational power has blossomed, convergent numerical and simulation techniques (see for example [2, 6, 7, 10, 28, 31, 41]) have become serious candidates for real time pricing, and an important issue is the trade off between the pricing accuracy and the computational cost.

Our approach will be to a construct general families of lower bounds for the price of the put option. We call the these families the M–point bounds for $M = 1, 2, \ldots$. The general M-point bound can be obtained recursively from the $(M-1)$-Point bound, and is expressed in terms of Normal integrals of dimension M or less. We will describe the 1 and 2-Point bounds and show the results of an extensive simulation. The M-point bound is parametric, depending on $3M-1$ parameters, and it is analytic modulo the need to compute the normal integrals, which can be effected very efficiently when $M = 1, 2$. Maximizing the bound with respect to the parameters, we obtain a tight lower bound on the price of the American put. This maximization is performed numerically and can be accomplished in under a second. In simulations, we compare the tightness of the lower bound with other methods. One of the virtues, as compared with data driven and simulation based methods is that the price output is guaranteed to be a lower bound.

Problem Formulation. We assume that the underlying asset undergoes a geometric Brownian motion, the risk neutral dynamics being given by

$$dS(t) = (r-\delta)S(t)dt + \sigma S(t)dW(t) \qquad (1)$$

where $dW(t)$ is a standard Wiener process, the risk free interest rate is r, δ is a constant dividend rate, and σ is the volatility parameter. Further, we assume that r, δ, σ are known *apriori*. Let $P(S, K, T, r, \delta, \sigma)$ be the fair value of the put option, where $S = S(0)$ is the current asset value, K is the strike price and T is the time to expiry (we will usually suppress the r, δ, σ dependence and write $P(S, K, T)$). An exercise function, $e(t) \le K$ with $e(0) < S(0)$, defines an exercise strategy as follows. If at time t, $S(t) \le e(t)$, then the strategy prescribes that the option be exercised (providing that it has not previously been exercised), yielding a profit of $K - e(t)$ at time t. The combination of a put option with an exercise function has a value that can be computed by taking the expected value of the discounted return under the risk neutral measure. More specifically, let τ be the time at which $S(t)$ first crosses the "barrier" $e(t)$ under the risk neutral dynamics. τ is a random variable that has some distribution, $f_\tau(t|e,S)$. This is conventionally called the first passage time distribution. The expected value of the discounted return is then given by

$$L(e|S,K,T)=\int_0^T dt\ f_\tau(t|e,S)e^{-rt}(K-e(t)) \qquad (2)$$

which depends on the particular choice of e. In fact for any choice of e, we know that this will be a lower bound on $P(S,K,T)$, thus,

$$P(S,K,T)\geq \sup_e L(e|S,K,T) \qquad (3)$$

If the sup is taken over a sufficiently general class of functions, then equality will result. Our approach will be to consider a class of functions for which the integral (2) can be computed. The class we will consider is the piecewise exponentials with M possible discontinuities at the times $\tau_1,\tau_2,\ldots,\tau_M$ where $\tau_M=T$. For convenience we set $\tau_0=0$,

$$e(t)=\begin{cases} e^{\alpha_i(t-\tau_{i-1})+\beta_i}, & \tau_{i-1}\leq t<\tau_i,\\ K, & t=T\end{cases} \qquad (4)$$

This is a $3M-1$ parameter class of functions, which we will denote the *M-point exponential functions*. Exponential exercise strategies have also been considered in [34]. There are some restrictions on the parameters, though. Since we wish to enforce $S\geq e(0)$, and $e(t)<K$ for $t<T$, we see that

$$\beta_1 \leq \min(\log S,\log K), \qquad \beta_i\leq \log K, \qquad (5)$$

$$\alpha_i \leq \frac{\log K-\beta_i}{\tau_i-\tau_{i-1}} \qquad (6)$$

The bounds resulting from the use of the M-point functions will be denoted the *M-point bounds*,

$$L_M(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}|S,K,T) \qquad (7)$$

When $M=1,2$ we will use the simpler notation $L(\alpha,\beta|S,K,T)$ and $L_2(\boldsymbol{\alpha},\boldsymbol{\beta},\tau|S,K,T)$.

The bounds we present are for the put option on a dividend paying stock, the price of the corresponding call option, $C(S,K,T,r,\delta,\sigma)$ can be computed using a put-call symmetry relationship, for example [14, 13],

$$C(S,K,T,r,\delta,\sigma)=P(K,S,T,\delta,r,\sigma). \qquad (8)$$

Using this relationship, the lower bound for put options is easily converted into a lower bound for call options.

2 RESULTS

We evaluate the integral in (2) for any $e(t)$ of the form given in (4). In so doing we compute a $3M-1$ parameter family of lower bounds on the price of the American put option, which we maximize to develop a computational approach to pricing the American put option. We provide in depth comparisons, via simulation, with some existing approaches. The family of M-point lower bounds is expressed in terms of the M-dimensional multivariate normal distribution function, defined by

$$N_M(\mathbf{x};\Sigma) = \int_{-\infty}^{x_1}dy_1\ldots\int_{-\infty}^{x_M}dy_M\ n_M(\mathbf{y};\Sigma) \qquad (9)$$

$$n_M(\mathbf{y};\Sigma) = \frac{1}{(2\pi)^{M/2}det\Sigma^{1/2}}e^{-\frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y}}$$

where $\mathbf{x}$ is an M-dimensional vector and Σ is an $M\times M$-dimensional symmetric correlation matrix ($\Sigma_{ii}=1$) that contains $\frac{1}{2}M(M-1)$ parameters. When $M=1$, $N_1(x)$ is conventionally written $N(x)$ and is related to the error function[1] by

$$N(x)=N_1(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}dy\ e^{-\frac{1}{2}y^2} = \frac{1}{2}\left[1+erf\left(x/\sqrt{2}\right)\right] \qquad (10)$$

When $M=2$, $N_2(\mathbf{x};\gamma)$ is conventionally called the bivariate normal integral given by

$$N(\mathbf{x};\gamma) = N_2(\mathbf{x};\gamma), \qquad (11)$$

$$= \int_{-\infty}^{x_1}ds\int_{-\infty}^{x_2}\frac{dt\ e^{-\frac{s^2-2\gamma st+t^2}{2(1-\gamma^2)}}}{2\pi\sqrt{1-\gamma^2}}, \qquad (12)$$

where $-1\leq\gamma\leq 1$. Since our results are expressed in terms of these functions, it is important to be able to compute them efficiently. N_1 and N_2 have been studied in some depth, and fast routines can be developed using the results in [16, 18, 20, 23, 42]. For convenience in notation, we will write

$$\aleph_T(x)=N\left(\frac{x}{\sqrt{\sigma^2T}}\right), \qquad \aleph_T(\mathbf{x},\gamma)=N_2\left(\frac{\mathbf{x}}{\sqrt{\sigma^2T}};\gamma\right) \qquad (13)$$

We present the bounds for $M=1$ and $M=2$. The proofs and full recursive computation of the general M-point bounds are postponed to a later exposition due to space constraints.

2.1 1-POINT BOUNDS

M=1, non-zero dividend rate. The final bound is given in the formula below.

$$P(S,K,T) \geq L(\alpha,\beta|S,K,T) = Ke^{-rT}J_1(\alpha,\beta|S,K,T) - Se^{-\delta T}J_2(\alpha,\beta|S,K,T) \qquad (14)$$

[1] $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x ds\ e^{-s^2}$

for all $\alpha,\ \beta$ that satisfy the constraints (6), with J_1 and J_2 given by

$$J_1(\alpha,\beta|S,K,T) = e^{-\frac{\rho l}{\sigma^2}+rT}\left[e^{-\frac{\lambda_1 l}{\sigma^2}}\aleph_T(\lambda_1 T - l)\right.$$
$$\left.+e^{\frac{\lambda_1 l}{\sigma^2}}\aleph_T(-\lambda_1 T - l)\right] - \aleph_T(d^-) + \aleph_T(\kappa^+)$$
$$+e^{-\frac{2l\rho}{\sigma^2}}\left[\aleph_T(d^- - 2l) - \aleph_T(\kappa^-)\right] \qquad (15)$$

$$J_2(\alpha,\beta|S,K,T) = e^{-\frac{l(\rho+\sigma^2)}{\sigma^2}+\delta T}\left[e^{-\frac{\lambda_2 l}{\sigma^2}}\aleph_T(\lambda_2 T - l)\right.$$
$$\left.+e^{\frac{\lambda_2 l}{\sigma^2}}\aleph_T(-\lambda_2 T - l)\right] - \aleph_T(d^+) + \aleph_T(\kappa^+ + \sigma^2 T)$$
$$+e^{-\frac{2l(\rho+\sigma^2)}{\sigma^2}}\left[\aleph_T(d^+ - 2l) - \aleph_T(\kappa^- + \sigma^2 T)\right] \qquad (16)$$

where $\rho(\alpha) = r - \delta - \frac{1}{2}\sigma^2 - \alpha$, $l(\beta) = \log S - \beta$, and the following definitions are used,

$$\begin{array}{l} \lambda_1(\rho) = \sqrt{\rho^2 + 2\sigma^2 r} \\ \lambda_2(\rho) = \sqrt{(\rho+\sigma^2)^2 + 2\sigma^2\delta} \\ d^{\pm} = \log\frac{S}{K} + (r - \delta \pm \frac{1}{2}\sigma^2) \end{array}, \qquad \kappa^{\pm} = \rho T \pm l \quad (17)$$

Notice that the dependence on α and β are solely though ρ and l respectively. Using the asymptotic form, $N(x) \to 1 - e^{-x^2/2}/x\sqrt{2\pi}$ as $x \to \infty$, we can deduce that $J_1 \to \aleph_T(-d^-)$ and $J_2 \to \aleph_T(-d^+)$ as α or β tend to $-\infty$ and thus, as expected, in this limit we recover the price of the European put option.

It is possible to consider certain special cases. The simplest is to set $\alpha = 0$, in which case the exercise boundary is a constant and we recover the price of the capped call option (using put-call symmetry) in [10]. Another is to require that α be chosen so that $e(T) = K$, requiring that $\alpha = (\log K - \beta)/T$. Algebraically, these restrictions do not lead to significant simplifications in the form for the bound, so we do not explicitly present the results here. Another simplification is to set $r = 0$. The form of the bound considerably simplifies in this regime, but the results are not interesting. When $T \to \infty$, we have the case of the perpetual American put option. In this case the bound becomes

$$P(S,K,\infty) \geq Ke^{-\frac{l(\lambda_1(\rho)+\rho)}{\sigma^2}} - Se^{-\frac{l(\lambda_2(\rho)+\rho+\sigma^2)}{\sigma^2}} \quad (18)$$

which we can optimize with respect to ρ and l. Since the optimal exercise boundary must be a constant (w.r.t.time), we can set $\alpha = 0$, in which case $\lambda_1 = \lambda_2$. Optimizing w.r.t. β, and denoting the optimal exercise price by S^*, we recover the price of the perpetual American put option (see for example [39]),

$$P(S,K,\infty) \quad = \quad \begin{cases} K - S & \mathcal{S} \leq S^* \\ (K - S^*)\left(\frac{S^*}{S}\right)^{\gamma} & S > S^* \end{cases} \quad (19)$$

where

$$\gamma = \frac{q + \sqrt{q^2 + 2r\sigma^2}}{\sigma^2},\ \ q = r - \delta - \frac{1}{2}\sigma^2,\ \ S^* = \frac{\gamma K}{1+\gamma}$$

This expression can be used to upper bound $P(S,K,T)$. Better upper bounds can be found in [11] and [15].

M=1, zero dividend rate. When the dividend rate is zero, $\delta = 0$, some simplifications occur. Since the zero dividend rate situation may be of particular use, we present the bounds in this case separately even though they are easily derivable from the general 1-point bound. The lower bound is given by

$$\begin{aligned} P(S,K,T) &\geq L(\alpha,\beta|S,K,T) \\ &= Ke^{-rT}\hat{J}_1(\alpha,\beta|S,K,T) \\ &\quad -S\hat{J}_2(\alpha,\beta|S,K,T) \end{aligned} \qquad (20)$$

for all $\alpha,\ \beta$ that satisfy the constraints (6), with $\hat{J}_1$ and $\hat{J}_2$ given by

$$\hat{J}_1(\alpha,\beta|S,K,T) = e^{-\frac{\rho l}{\sigma^2}+rT}\left[e^{-\frac{\lambda l}{\sigma^2}}\aleph_T(\lambda T - l)\right.$$
$$\left.+e^{\frac{\lambda l}{\sigma^2}}\aleph_T(-\lambda T - l)\right] - \aleph_T(d^-) + \aleph_T(\kappa^+)$$
$$+e^{-\frac{2l\rho}{\sigma^2}}\left[\aleph_T(d^- - 2l) - \aleph_T(\kappa^-)\right] \qquad (21)$$

$$\hat{J}_2(\alpha,\beta|S,K,T) = \aleph_T(-d^+) + e^{-\frac{2l(\rho+\sigma^2)}{\sigma^2}}\aleph_T(d^+ - 2l) \qquad (22)$$

where $\rho = r - \frac{1}{2}\sigma^2 - \alpha$, $l = \log S - \beta$, and the following definitions are used,

$$\begin{array}{ll} \lambda(\rho) = \sqrt{\rho^2 + 2\sigma^2 r}, & \kappa^{\pm} = \rho T \pm l, \\ d^{\pm} = \log\frac{S}{K} + (r \pm \frac{1}{2}\sigma^2)T. & \end{array} \qquad (23)$$

When $T \to \infty$, (19) still holds in the zero dividend case, with $\gamma = 2r/\sigma^2$.

2.2 RECURSIVE COMPUTATION OF THE M-POINT BOUNDS

Suppose that the n-point bound $L_n(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}|S,K,T)$ has been computed for arbitrary S, K, T and any n-dimensional vectors $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}$ that satisfy the constraints (6). The $(n+1)$-point exponential exercise strategy that yields an $(n+1)$-point bound can be viewed as a two stage process. Exercise with respect to the first section of the boundary up to τ_1. If by this time, exercise has not occurred, then exercise with respect to the remaining n point boundary, and in this case, the n-point bound (averaged over the price at τ_1) gives the value. This is the intuition behind the recursive approach that gives the $(n+1)$-point bound in terms of the n-point bound. This approach is easy to automate, though we do not give the details here. Since we have the 1-point bound, we can compute the 2-point bound, and so on. Unfortunately, things are not that rosy. During the course of the recursion, to compute the M-point bound, it becomes necessary to compute multivariate normal integrals up to dimension M. When $M = 2$, however, the bivariate normal integral can be computed very efficiently. The 2-point bound that results from the recursion is quite cumbersome, involving summations over Normal integrals. However, the important point is that given the 1-Point bound, a completely automated, recursive prescription generates the two point bound. We present the formulas thus generated here, and we postpone the discussion of the details behind the recursive prescription.

2-Point Bounds. The 2-point exponential function is specified once the parameters, $\alpha_1, \alpha_2, \beta_1, \beta_2, \tau$ are specified (that satisfy (6)). Define the functions a_j, b_j, c_j, d_j for $j = 1, \dots, 12$ as given in the following tables.

j	$a_j(\alpha,\beta,T)$	$b_j(\alpha,\beta,T)$	$d_j(\alpha,\beta,T)$
1	$Ke^{\frac{\beta(\rho+\lambda_1)}{\sigma^2}}$	$-\frac{\rho+\lambda_1}{\sigma^2}$	$-1/\sqrt{\sigma^2 T}$
2	$Ke^{\frac{\beta(\rho-\lambda_1)}{\sigma^2}}$	$\frac{\lambda_1-\rho}{\sigma^2}$	$-1/\sqrt{\sigma^2 T}$
3	$-Ke^{-rT}$	0	$1/\sqrt{\sigma^2 T}$
4	Ke^{-rT}	0	$1/\sqrt{\sigma^2 T}$
5	$Ke^{-rT+\frac{2\beta\rho}{\sigma^2}}$	$-\frac{2\rho}{\sigma^2}$	$-1/\sqrt{\sigma^2 T}$
6	$-Ke^{-rT+\frac{2\beta\rho}{\sigma^2}}$	$-\frac{2\rho}{\sigma^2}$	$-1/\sqrt{\sigma^2 T}$
7	$-e^{\frac{\beta(\rho+\sigma^2+\lambda_2)}{\sigma^2}}$	$-\frac{\rho+\lambda_2}{\sigma^2}$	$-1/\sqrt{\sigma^2 T}$
8	$-e^{\frac{\beta(\rho+\sigma^2-\lambda_2)}{\sigma^2}}$	$\frac{\lambda_2-\rho}{\sigma^2}$	$-1/\sqrt{\sigma^2 T}$
9	$e^{-\delta T}$	1	$1/\sqrt{\sigma^2 T}$
10	$-e^{-\delta T}$	1	$1/\sqrt{\sigma^2 T}$
11	$-e^{-\delta T+\frac{2\beta(\rho+\sigma^2)}{\sigma^2}}$	$-\frac{2\rho}{\sigma^2}-1$	$-1/\sqrt{\sigma^2 T}$
12	$e^{-\delta T+\frac{2\beta(\rho+\sigma^2)}{\sigma^2}}$	$-\frac{2\rho}{\sigma^2}-1$	$-1/\sqrt{\sigma^2 T}$

j	$\sqrt{\sigma^2 T}c_j(\alpha,\beta,T)$
1	$\lambda_1 T+\beta$
2	$-\lambda_1 T+\beta$
3	$-\log K+(r-\delta-\frac{1}{2}\sigma^2)T$
4	$\rho T-\beta$
5	$-\log K+(r-\delta-\frac{1}{2}\sigma^2)T+2\beta$
6	$\rho T+\beta$
7	$\lambda_2 T+\beta$
8	$-\lambda_2 T+\beta$
9	$-\log K+(r-\delta+\frac{1}{2}\sigma^2)T$
10	$(\rho+\sigma^2)T-\beta$
11	$-\log K+(r-\delta+\frac{1}{2}\sigma^2)T+2\beta$
12	$(\rho+\sigma^2)T+\beta$

where,$\rho = r-\delta-\frac{1}{2}\sigma^2-\alpha$, $l = \log S-\beta$ and λ_1, λ_2 are defined in (17). Using this notation, one can rewrite the 1-Point bound in the following convenient format.

$$L_1(\alpha,\beta,|S,K,T) = \sum_j a_j e^{b_j \log S} N(c_j + d_j \log S). \tag{24}$$

Define correlation coefficients for $j = 1,\dots,12$ by $\gamma_j = sign(\hat{d}_j)\sqrt{\tau/T}$, and define the functions $\hat{a}_j, \hat{b}_j, \hat{c}_j, \hat{d}_j$ by

$$\begin{array}{ll} \hat{a}_j = a_j(\alpha_2,\beta_2,T-\tau) & \hat{b}_j = b_j(\alpha_2,\beta_2,T-\tau) \\ \hat{c}_j = c_j(\alpha_2,\beta_2,T-\tau) & \hat{d}_j = d_j(\alpha_2,\beta_2,T-\tau) \end{array} \tag{25}$$

where the functions $a_j(\cdot)$, $b_j(\cdot)$, $c_j(\cdot)$, $d_j(\cdot)$ are given in the tables above. The 2-Point bound is then given by

$$\begin{aligned} L_2(\boldsymbol{\alpha},\boldsymbol{\beta},\tau|S,K,T) &= I_1(\alpha_1,\beta_1|S,K,\tau) \\ &+\hat{I}_2(\alpha_1,\beta_1,\beta_2|S,K,\tau)\cdot[[\beta_2>\beta_1+\alpha\tau]] \\ &+e^{-r\tau}\sum_{j=1}^{12}\hat{a}_j e^{(\hat{b}_j(\rho_1+\alpha_1)+\frac{1}{2}\hat{b}_j^2\sigma^2)\tau+\hat{b}_j\log S}N(\mathbf{v}_j;\gamma_j) \\ &-e^{-r\tau-\frac{2l_1\rho_1}{\sigma^2}}\times \\ &\sum_{j=1}^{12}\hat{a}_j e^{(\hat{b}_j(\rho_1+\alpha_1)+\frac{1}{2}\hat{b}_j^2\sigma^2)\tau+\hat{b}_j(2\beta_1-\log S)}N(\mathbf{w}_j;\gamma_j) \end{aligned} \tag{26}$$

where $[[expr]]$ evaluates to 1 when $expr$ is true and zero otherwise. $\rho_1 = r-\delta-\frac{1}{2}\sigma^2-\alpha_1$, $l_1 = \log S-\beta_1$, and γ_j are the correlations defined earlier. $\mathbf{v}_j$, $\mathbf{w}_j$ are given by

$$\mathbf{v}_j = \begin{bmatrix} \sqrt{\frac{T-\tau}{T}}\left(\hat{c}_j+\hat{d}_j(\eta+\hat{b}_j\sigma^2\tau)\right) \\ \frac{-A+(\rho_1+\hat{b}_j\sigma^2)\tau}{\sqrt{\sigma^2\tau}} \end{bmatrix} \tag{27}$$

$$\mathbf{w}_j = \begin{bmatrix} \sqrt{\frac{T-\tau}{T}}\left(\hat{c}_j+\hat{d}_j(\eta+\hat{b}_j\sigma^2\tau-2l_1)\right) \\ \frac{-A+(\rho_1+\hat{b}_j\sigma^2)\tau-2l_1}{\sqrt{\sigma^2\tau}} \end{bmatrix} \tag{28}$$

where $\eta = \log S+(\rho_1+\alpha_1)\tau$ and $A = \beta_1-\log S+[\beta_2-\beta_1-\alpha_1\tau]^+$. Finally, I_1 and $\hat{I}_2$ are given by

$$\begin{aligned} I_1(\alpha,\beta|S,K,T) &= Ke^{-\frac{\rho l}{\sigma^2}}\Big[e^{-\frac{\lambda_1 l}{\sigma^2}}\aleph_T(\lambda_1 T-l) \\ &\quad+e^{\frac{\lambda_1 l}{\sigma^2}}\aleph_T(-\lambda_1 T-l)\Big] \\ &-Se^{-\frac{l(\rho+\sigma^2)}{\sigma^2}}\Big[e^{-\frac{\lambda_2 l}{\sigma^2}}\aleph_T(\lambda_2 T-l) \\ &\quad+e^{\frac{\lambda_2 l}{\sigma^2}}\aleph_T(-\lambda_2 T-l)\Big] \end{aligned} \tag{29}$$

$$\begin{aligned} \hat{I}_2(\alpha_1,\beta_1,\beta_2|S,K,\tau_1) &= Ke^{-r\tau_1}\Big(-\aleph_{\tau_1}(\hat{d}^-) \\ &+\aleph_{\tau_1}(\kappa^+)+e^{-\frac{2l_1\rho_1}{\sigma^2}}\Big[\aleph_{\tau_1}(\hat{d}^--2l_1)-\aleph_{\tau_1}(\kappa^-)\Big]\Big) \\ &-Se^{-\delta\tau_1}\Big(-\aleph_{\tau_1}(\hat{d}^+)+\aleph_{\tau_1}(\kappa^++\sigma^2\tau_1) \\ &+e^{-\frac{2l_1(\rho_1+\sigma^2)}{\sigma^2}}\Big[\aleph_{\tau_1}(\hat{d}^+-2l_1)-\aleph_{\tau_1}(\kappa^-+\sigma^2\tau_1)\Big]\Big) \end{aligned} \tag{30}$$

where $\rho_1 = r-\delta-\frac{1}{2}\sigma^2-\alpha_1$, $l_1 = \log S-\beta_1$, $\hat{d}^\pm = \log S-\beta_2+(r-\delta\pm\frac{\sigma^2}{2})\tau_1$, and $\kappa^\pm = \rho_1\tau_1 \pm l_1$.

We have taken this much effort to merely state the 2-Point bound. Once again we emphasize that the bound is complex, however, automatically generated.

2.3 OPTIMIZING THE M-POINT BOUNDS

We illustrate the procedure on the 1-Point bounds for simplicity. We obtain the tighter lower bound by maximizing

(14) with respect to α and β, subject to the constraints (6). Define

$$(\alpha^*, \beta^*) = \operatorname*{argmax}_{\alpha,\beta} L(\alpha, \beta | S, K, T, r, \sigma) \tag{31}$$

The lower bound will then be $L(\alpha^*, \beta^* | S, K, T, r, \sigma)$. Unfortunately, setting the partial derivatives of $L(\cdot)$ equal to zero to obtain (α^*, β^*) did not yield analytically, so we proceed numerically. Since most optimization algorithms are gradient based, it is necessary to compute ∇J_1 and ∇J_2. From (17), we see that $\frac{\partial}{\partial\alpha} = -\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial\beta} = -\frac{\partial}{\partial l}$, and thus it suffices to compute $\frac{\partial L}{\partial\rho}$ and $\frac{\partial L}{\partial l}$. These partial derivatives can be computed directly from (15) and (16), and we do not explicitly compute them here.

The constraints (6) create some complications in the optimization. A simpler approach is to write

$$(\alpha, \beta) = (f_1(u, v), f_2(u, v)), \tag{32}$$

where u, v are now unconstrained real parameters, and the functions f_1, f_2 are chosen so as to incorporate the constraints on α, β. Thus $L(\alpha, \beta)$ is implicitly a function of u, v and we can optimize L with respect to the unconstrained variables u, v. By the chain rule, the partial derivatives of L with respect to u, v can be computed using the formula

$$\frac{\partial L}{\partial u} = L_\alpha \frac{\partial f_1}{\partial u} + L_\beta \frac{\partial f_2}{\partial u} \tag{33}$$

$$\frac{\partial L}{\partial v} = L_\alpha \frac{\partial f_1}{\partial v} + L_\beta \frac{\partial f_2}{\partial v} \tag{34}$$

where

$$L_\alpha = \left.\frac{\partial L}{\partial \alpha}\right|_{\substack{\alpha = f_1(u,v) \\ \beta = f_2(u,v)}} \quad \text{and} \quad L_\beta = \left.\frac{\partial L}{\partial \beta}\right|_{\substack{\alpha = f_1(u,v) \\ \beta = f_2(u,v)}} \tag{35}$$

and since L_α and L_β have can be computed from (14), the gradient with respect to the unconstrained u, v can also be computed, hence any gradient based optimization technique can now be used to maximize $L(u, v)$, for example [4, chapter 7].

The particular functions (f_1, f_2) that we chose were

$$\alpha = f_1(u, v) = \frac{\log K - f_2(u, v)}{T(1 + e^{-u})} \tag{36}$$

$$\beta = f_2(u, v) = \begin{cases} \log S - v^2 & S \le K \\ \log K - v^2 & S > K \end{cases} \tag{37}$$

In addition to enforcing the constraints (6), these transformations also enforce the constraint that $\alpha \geq 0$, which is not necessary for the validity of the bound, but should hold from the financial perspective. Formally the pricing algorithm is to maximize $L(u, v)$. The resulting value of L is the final lower bound on $P(S, K, T, r, \sigma)$, and further, the optimal u, v imply an exercise boundary given by $e(t) = e^{f_1(u,v)t + f_2(u,v)}$.

3 SIMULATIONS

We tested three bounds, namely the 0-Point bound (equivalent to the one in [10]), where the exercise strategy was a constant (one free paramenter), the 1-Point bound (two free parameters), and the 2-Point bound (five free parameters). In each of the cases, the lower bound was maximized with respect to the parameters in their allowed ranges. This constrained optimization was effected using the trick in (32). The optimization algorithm we used was the Nelder–Mead interior point method that is available as a canned software program (for example on MATLAB this is the `fminsearch` function). Our goal was not to specialize the optimization algorithm in any way, but rather demonstrate our methods with respect to a canned optimization algorithm. The 0-Point and 1-Point bounds are relatively easy to optimize and so we optimized them till the parameters and lower bound converged to within tolerance of $tolX$ and $tolB$ respectively. For the 0-Point bound, we used $tolX = 10^{-8}$ and $tolB = 10^{-8}$. For the 1-Point we used tolerances of $tolX = 10^{-3}$ and $tolB = 10^{-5}$. For the 2-Point bound, we used $tolX = 10^{-4}$ and $tolB = 10^{-5}$, however, since the optimization problem is quite complex, a natural trade off appears. The better the optimization, the better the bound, however, the longer it takes. Thus, we use the optimization algorithm with diferent numbers of iterations, to explore in greater detail this trade-off – we implemented runs with $25, 50, 100, 200, 500$ iterations. The optimization algorithm did not use gradient information, so the speed of our algorithm could be considerably enhanced this way, however we do not pursue these issues here as they are largely implementational in nature. Further, there are numerical instabilities that may be encountered in computing the bound as it involves the differences between exponentials where the exponents can be quite large. Robust numerical techniques need to be used to ensure that the bounds are computed accurately, otherwise the optimization will severely fail to converge. The complete discussion of such numerical issues is postponed.

We constructed a test bed of about 130,000 put options where $(S, K, T, r, \delta, \sigma)$ were generated randomly[2]. S was chosen uniformly from the range $[50, 100]$, and conditional upon S, K was chosen uniformly in the range $[0.8S, 1.2S]$. Independent of S, K and each other, T, r, δ, σ were each generated uniformly in the ranges $T \in [0.1, 5]$, $r \in [0, 0.1]$, $\delta \in [0, 0.1]$, $\sigma \in [0.1, 0.6]$. We used a convergent binomial tree method with a small enough time step as a proxy for the true price. If N is the number of time steps used then the compute time of a binomial tree method is $\Theta(N^2)$ and the accuracy is $\Theta(N^{-1})$. We pick $N = 15000$ to compute the "true" price of the option, which is the number of steps suggested in [10]. This gives a reasonable proxy for the true price. To compare the $0, 1, 2$-Point bounds with existing methods, we implemented the Binomial tree method with time steps ranging from 20 to 1500. By transitivity, the many methods in the literature can be compared with out method through how they perform with respect to the binomial tree method. An extensive comparison of several such methods was performed in [10], and so we do not explicitly perform such a comparison here. Our results are shown in

[2] For the purposes of reproducing these results, this data set is available through the author's web site or via email.

Figure 1 below. We use the RMS relative error as a comparison metric,

$$RMS_{rel} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\hat{P}_i - P_i}{P_i}\right)^2}, \tag{38}$$

where P_i is the true price, and $\hat{P}_i$ is the predicted price. From the investor's perspective, one is usually interested in the percentage return, hence this is the appropriate metric. There are a number of ways to interpret this graph. The binomial trees with the different number of steps are shown with square points. The line fitting these points (with slope -2) allows one to compare methods with the binomial trees. An algorithm which appears below the line is better than a binomial tree in the following sense. Drawing the vertical up to the binomial tree line, we obtain the amount of additional computation the binomial tree uses to achieve the same accuracy. Drawing the rightward horizontal to the binomial tree line, we obtain the drop in accuracy the binomial tree suffers if restricted to the same amount of computation as this algorithm. To illustrate, consider the 2-Point bound with 50 iterations of optimization. On average, the number of floating point operations required was about $10^{6.1}$, to achieve an accuracy of $RMS_{rel} = 10^{-3.7}$. To put this in perspective with mordern computing facilities, a typical machine running at 2 gigahertz executes a single floating point operation in about 200 clock cycles which equates to about 10^7 flops per second, thus the 2-Point bound with 50 iterations would take under a second to compute. The binomial tree with comparable accuracy uses about $10^{7.5}$ flops, a factor of 25 slower. The binomial tree with comparable runtime uses about 400 steps, and achieves an accuracy $RMS_{rel} = 10^{-2.95}$, almost an order of magnitude worse. Thus we conclude that the 2-Point bound with 50 iterations has an accuracy comparable to a 2000 step binomial tree, and a computational cost comparable to a 400 step binomial tree. We can further analyse these results by looking separately at the options with price ≤ 1 and those with price > 1. These results are shown in Figure 2. For options with large

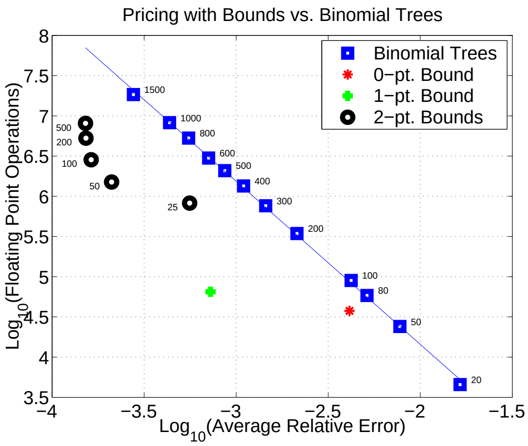
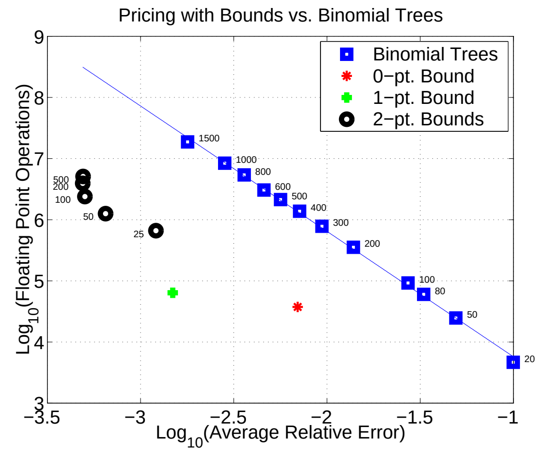
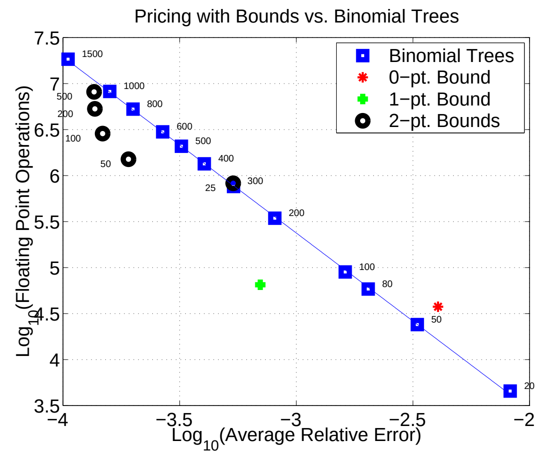


Figure 1: Binomial trees vs. the M-Point bounds.



(a)True Price ≤ 1

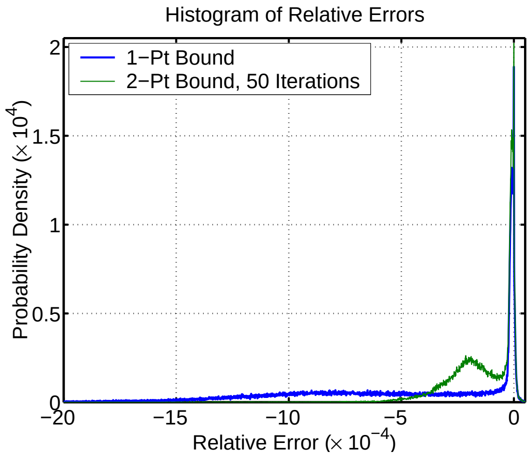


(b)True Price > 1

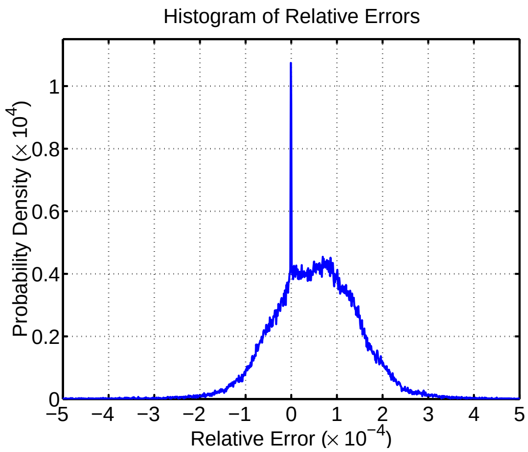
Figure 2: Relative errors for options priced (a) below 1 and (b) above 1.

prices, notice that the algorithms perform better. In this case the 2-Point bound with 50 iterations delivers the accuracy of an 800 step binomial tree using computation equivalent to a 400 step binomial tree. The big bonus is when the option price is small, in which case, the binomial methods degenerate considerably, however our bounds are quite robust. When the option price is small, the 2-Point bound with 50 iterations delivers the accuracy of a 4000 step binomial tree using computation equivalent to a 400 step binomial tree!

The distribution of the relative errors is given in the figures below. Notice that the lower bounds are not strictly less than 0. This phenomenon is due to a combination of numerical inaccuracies, and the fact that the proxy that we use for the price, namely the binomial tree with 15000 steps, is not the true price – this proxy becomes worse when the option price is small. However, in general, the lower bound is performing as it should. The binomial tree relative errors display a slight positive bias, but otherwise are somewhat random. An option that is cheaper than the lower bound is clearly cheap, however a similar statement cannot be made for the binomial tree price.



(a)Lower bounds.



(b)1500 step binomial tree.

Figure 3: Distribution of the relative errors.

4 CONCLUSIONS

We have presented families of lower bounds for the price of the American option. Our bound performs well when compared with binomial trees. Our bounds are robust, and efficient, as has been demonstrated in the simulations. In addition our bound simultaneously furnishes an exponential exercise strategy. An improved strategy can be computed by determining the value of the stock price at which the bound equals the exercise value. This curve is the instantaneous exercise strategy implied by the bound.

A number of issues have been glossed over, and some have been postponed. These include the derivations, and the full discussion of the numerical techniques involved. We have implemented canned optimization algorithms that do not use gradient information, so by improving the optimization, one should be able to get a factor of two or three improvement in efficiency.

Among the other interesting questions that one could ask, one is how the Greeks derived from the bounds would compare to the true Greeks? The Greeks are important for hedging purposes and are sometimes just as useful as the price itself. We postpone the discussion of the Greeks to a later occasion. As was introduced in [10], one can use bounds to derive better approximations by using data driven techniques to predict the discrepancy between the bound and the true price. The bound used in [10] was the 0-Point bound, in the form of the capped call option. The 1-Point bound is considerably more accurate, and can be computed with only slightly more cost. Thus data driven approximations using these improved bounds as a starting point are likely to be much more accurate. We will discuss these issues in greater detail at another time.

Future directions include the analysis of how the accuracy of the M-Point bounds depends on M. Further, efficient techniques for computing the Normal integrals are needed to compute the M-Point bounds for larger M and this is an active area of research.

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