An Analysis of the Maximum Drawdown Risk Measure

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Introduction.

The maximum cumulative loss from a peak to a following bottom, commonly denoted the *maximum drawdown MDD*, is a measure of how sustained one's losses can be. Large drawdowns usually lead to fund redemptions, and so the MDD is the risk measure of choice for many money management professionals – a reasonably low MDD is critical to the success of any fund. Related to the MDD is the *Calmar ratio*[1], a risk adjusted measure of performance, that is given by the formula

$$\text{Calmar}(T) = \frac{\text{Return over } [0,T]}{MDD \text{ over } [0,T]}.$$

The *Sharpe ratio* is similar in that it is also a risk adjusted measure of performance, however the MDD risk measure is replaced by the standard deviation of the returns over intervals of size T.

The "square-root-T-law" is a well known law prescribing how the unnormalized Sharpe ratio scales with time. This law allows one to scale the Sharpe ratio so that comparing different systems is possible even when their Sharpe ratios were computed using *different* values of T. On the other hand, such similar scaling laws for the Calmar ratio are not known. As a result, the common practice is to compare Calmar ratios for portfolios over equal length time intervals (the typical choice is three years). Such a constraint on the use of the use of Calmar ratio is artificial, and, based upon the results that we will present, unnecessary.

Another task that is important for fund managers is the ability to construct portfolios that are optimal with respect to the risk adjusted performance. When the performance measure used is the Sharpe ratio, this leads to mean-variance portfolio analysis. A similar approach to portfolio optimization using the Calmar ratio as a criterion is not prevalent

[1]Similar to the Calmar ratio is the *Sterling* ratio, $\text{Sterling}(T) = \frac{\text{Return over } [0,T]}{MDD \text{ over } [0,T]-10\%}$, and our discussion applies equally well to the Sterling ratio.

primarily due to a lack of an analytical understanding regarding how the MDD of a portfolio is related to performance characteristics of the individual instruments.

In this article, we present analytical results relating the expected MDD to the mean return and the standard deviation of the returns. The detailed mathematical derivations are given in [10]. We also present formulas that relate the Calmar ratio to the Sharpe ratio. We introduce the *Normalized Calmar Ratio* which can be immediately compared for two portfolios. We also present some plots illustrating some of the portfolio aspects of the MDD, in particular, how the correlation factors in. Among our interesting findings is that an instrument with a negative return can be beneficial from the Calmar ratio point of view, if it is sufficiently uncorrelated.

Related Work. The drawdown at time t has been studied, and its distribution can be obtained analytically from the joint density of the maximum and the close of a Brownian motion (see for example [8]). Most work on the maximum drawdown is empirical in nature (for example [3, 4, 7, 12]). The most relevent theoretical result is for the case of a Brownian motion with zero drift, in which case, the full distribution of the maximum drawdown is given in [6]. Since we wish to relate the MDD to the drift, we cannot assume that the drift is zero. Portfolio optimization using the drawdown has also been considered in [5].

The Expected Maximum Drawdown.

Assume that the value of a portfolio follows a Brownian motion:

$$dx = \mu dt + \sigma dW \qquad 0 \leq t \leq T,$$

where time is measured in years, and μ is the average return per unit time, σ is the standard deviation of the returns per unit time and dW is the usual Wiener increment. This model assumes that profits are not reinvested. If profits are reinvested, then a Geometric Brownian motion is the appropriate model,

$$ds = \hat{\mu} s dt + \hat{\sigma} s dW \qquad 0 \leq t \leq T.$$

For such a case, equivalent formulas can be obtained by taking a log transformation: if $x = \log s$, then x follows a Brownian motion with $\mu = \hat{\mu} - \frac{1}{2}\hat{\sigma}^2$ and $\sigma = \hat{\sigma}$. (The MDD in this case is defined with respect to the percentage drawdown rather than absolute drawdown.) If the portfolio value follows a more complicated process, then the results for the Brownian motion can be used as benchmark.

Using results on the first passage time of a reflected Brownian motion, we find that the expected MDD has drastically different behavior according to whether the portfolio is profitable, breaking even or losing money. This "phase shift" in the behavior is highlighted by the asymptotic ($T \to \infty$) behavior in the formulas below. The asymptotic behavior is important because most trading desks are interested in long term performance, i.e.,

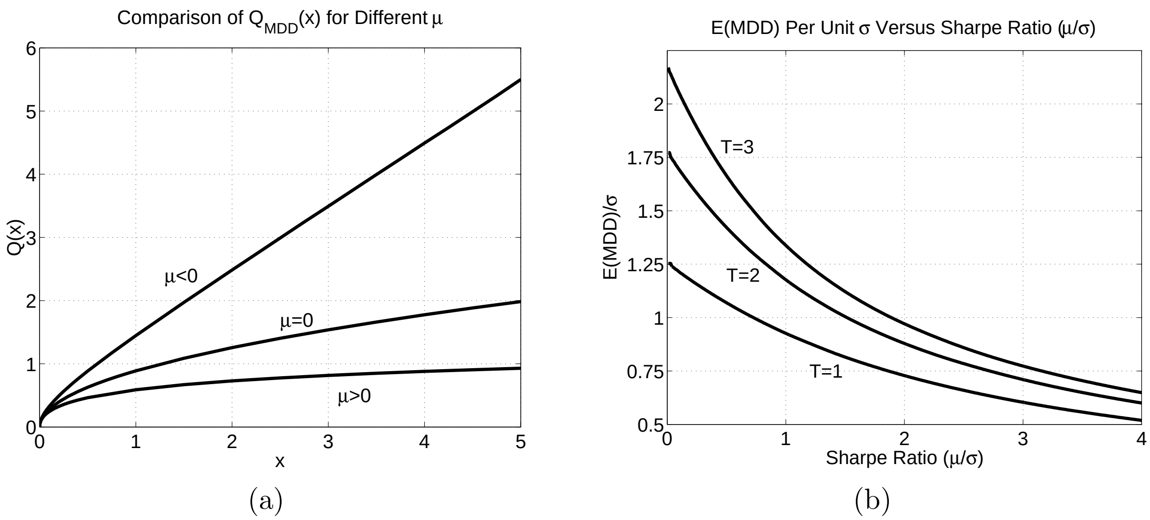

Figure 1: In (a) we show the behavior of the $Q_p(x)$, $Q_n(x)$ and the equivalent function for $\mu = 0$, illustrating the behavior of these functions for different μ regimes. In (b) we show how the expected MDD per unit variance depends on the Sharpe ratio for different values of T.

systems that can survive over the long run, with superior return and small drawdowns. The expression for the expected MDD is:

$$E(MDD) = \begin{cases} \frac{2\sigma^2}{\mu} Q_p\left(\frac{\mu^2 T}{2\sigma^2}\right) \overset{T\to\infty}{\longrightarrow} \frac{\sigma^2}{\mu}\left(0.63519 + 0.5\log T + \log\frac{\mu}{\sigma}\right) & \text{if } \mu > 0 \\ 1.2533\sigma\sqrt{T} & \text{if } \mu = 0 \\ \frac{-2\sigma^2}{\mu} Q_n\left(\frac{\mu^2 T}{2\sigma^2}\right) \overset{T\to\infty}{\longrightarrow} -\mu T - \frac{\sigma^2}{\mu} & \text{if } \mu < 0 \end{cases}$$

As can be noted, the scaling of the expected MDD with T undergoes a phase transition from T to $\sqrt{T}$ to $\log T$ as μ transitions from negative to zero to positive. One immediate use of this behavior is as a hypothesis test to determine if a portfolio is profitable, even or losing. The functions $Q_n(x)$ and $Q_p(x)$ are complicated integral expansions that do not have a convenient analytical form. They are independent of μ, σ and T, and so they are "universal functions" in the sense that they can be evaluated once and tabulated for future use. Such a table is given in [10] and can also be downloaded from [2]. Figure 1(a) shows the functions $Q_p(x)$ and $Q_n(x)$. The exact functional form including the distribution of the MDD, as well as a tabulation of values can be found in [10, 9]. From now on, we focus on the more interesting case of profitable ($\mu > 0$) portfolios. The discussion can easily be extended to all three regimes of μ.

Define the $\sqrt{T}$-scaled Sharpe ratio of expected performance by $\mathsf{Shrp} = \mu/\sigma$. The ex-

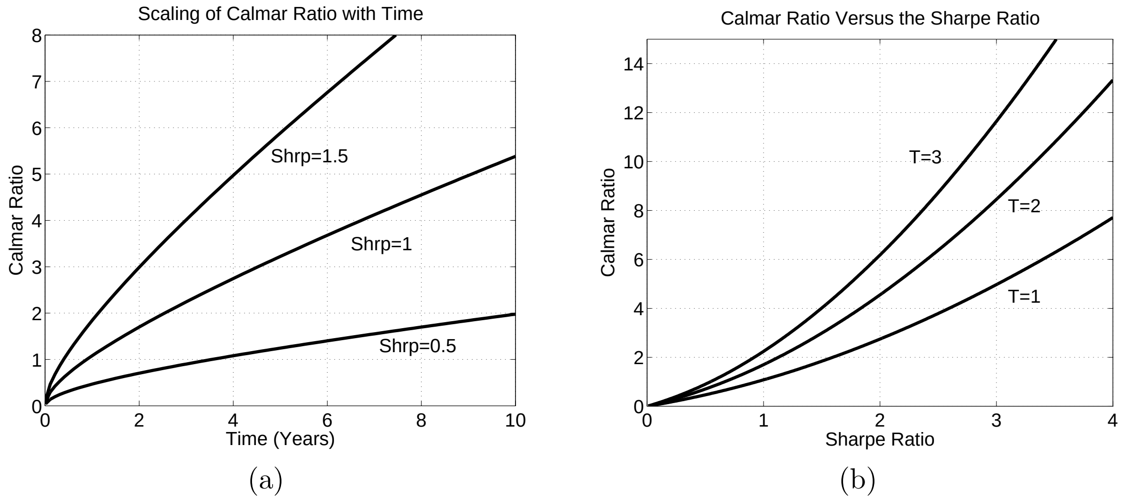

Figure 2: How Clmr depends on T and Shrp. In (a) we illustrate the how Clmr scales with time for different portfolio characteristics, and in (b) how it scales with Shrp at different times.

pected MDD normalized per unit of σ can be written entirely in terms of Shrp:

$$\frac{E(MDD)}{\sigma} = \frac{2Q_p\left(\frac{T}{2}\mathsf{Shrp}^2\right)}{\mathsf{Shrp}}$$

Figure 1(b) illustrates the dependence of $E(MDD)$ normalized per unit of σ on the Sharpe ratio, Shrp.

The Normalized Calmar Ratio.

First, we will deduce a relationship between the Sharpe ratio and the Calmar ratio. Consider the Calmar ratio of expected performance, Clmr, given by

$$\mathsf{Clmr}(T) = \frac{\mu T}{E(MDD)}.$$

Substituting this definition into the expression for $E(MDD)$, we get that

$$\mathsf{Clmr}(T) = \frac{\frac{T}{2}\mathsf{Shrp}^2}{Q_p\left(\frac{T}{2}\mathsf{Shrp}^2\right)} \overset{T\to\infty}{\longrightarrow} \frac{T\mathsf{Shrp}^2}{0.63519 + 0.5\log T + \log \mathsf{Shrp}} \tag{1}$$

Some interesting points to note are that the Calmar depends on μ and σ only through the scaled Sharpe ratio (the dependence of Clmr on T and on the normalized Sharpe ratio are illustrated in Figure 2); for fixed μ, σ, Clmr is increasing with T. Thus, knowing the Calmar

Portfolio	$\mu(\%)$	$\sigma(\%)$	Calmar	Time Interval (yrs)	Relative Strength
Π_1	25	10	5	[0,1]	1.00
Π_2	30	10	6	[0.5,2]	0.97
Π_3	25	12.5	6	[0,2]	0.64

Table 1: Some example portfolios.

ratio of a portfolio without knowing T is useless. If fund X has a Calmar of 5 and fund Y has a Calmar of 6, it is not clear which is a better fund. In fact it is possible that fund X is better! To make a better comparison, it is *necessary* to know the time intervals over which each Calmar ratio was computed, and scale appropriately. However, perhaps we can remove this dependence on T by standardizing the way the Calmar ratio is quoted. This can be accomplished by *normalizing* the Calmar ratio. More specifically, whenever a Calmar ratio is quoted, one should automatically incorporate the appropriate scaling so that the comparison becomes seamless. Despite how prevalent the MDD is as a measure of risk, such a systematic approach is not usually used, because the appropriate scaling behavior was not known. Our results provide exactly the necessary scaling behavior.

Fix a reference time frame τ (for example $\tau = 1$). If all Calmar ratios were quoted on this time frame, then comparing portfolios would be easy. For a given portfolio, suppose we have computed Shrp. In this case, from (1), for the time interval τ, we know that Clmr is expected to be $\mathsf{Clmr}(\tau) = \frac{\tau}{2}\mathsf{Shrp}^2/Q_p(\frac{\tau}{2}\mathsf{Shrp}^2)$. Similarly, at time T, we know that $\mathsf{Clmr}(T) = \frac{T}{2}\mathsf{Shrp}^2/Q_p(\frac{T}{2}\mathsf{Shrp}^2)$, and so to get the τ-*normalized*-Calmar ratio, we need to scale by a normalizing factor,

$$\gamma_\tau(T, \mathsf{Shrp}) = \frac{\frac{1}{T}Q_p(\frac{T}{2}\mathsf{Shrp}^2)}{\frac{1}{\tau}Q_p(\frac{\tau}{2}\mathsf{Shrp}^2)}$$

More specifically, if everyone agrees on the base time scale τ, then having computed the Calmar ratio, and μ, σ for a portfolio over the interval $[0, T]$, the τ-normalized-Calmar ratio $\overline{Calmar}(\tau)$ is given by

$$\overline{Calmar}(\tau) = \gamma_\tau(T, \mathsf{Shrp}) \times \text{Calmar ratio}.$$

Following the convention applied to quoting the Sharpe ratio, we suggest fixing the base time scale τ to one year.

Example: The idea is best illustrated by an example. Suppose that three portfolios Π_1, Π_2, Π_3 have the P&L statistics over their respective time intervals as illustrated in Table 1. How do we compare these portfolios if our criterion is the Calmar ratio? First, let us illustrate some of the intuition. If we compute Clmr for Π_1, we get roughly 3.8. Since its actual Calmar is higher, Π_1 seems to have negative autocorrelation for its returns, i.e., it seems to be outperforming. Similarly, $\mathsf{Clmr}(\Pi_2) = 6.76$ and $\mathsf{Clmr}(\Pi_3) = 4.55$. It seems that Π_2 is underperforming and is the worst, however it is not clear how to compare Π_1 with Π_3 at this point. By computing the normalized (to $\tau = 1$) Calmar ratios, we will be in a better

Fund	$\mu(\%)$	$\sigma(\%)$	T(yrs)	MDD	$Calmar$	**E[MDD]**	$\overline{\textbf{Calmar}}$	$\boldsymbol{\beta}$
$S\&P500$	10.04	15.48	24.25	46.28	5.261	**44.56**	**0.6104**	**1**
$FTSE100$	7.01	16.66	19.83	48.52	2.865	**55.54**	**0.4395**	**0.5003**
$NASDAQ$	11.20	24.38	19.42	75.04	2.899	**77.87**	**0.4402**	**0.5407**
DCM	15.65	5.78	3.08	3.11	15.50	**4.770**	**6.541**	**27.76**
NLT	3.35	16.03	3.08	25.40	0.4062	**31.35**	**0.2202**	**0.1331**
OIC	17.19	4.52	1.16	0.42	47.48	**2.493**	**42.31**	**212.0**
TGF	8.48	9.83	4.58	8.11	4.789	**15.84**	**1.752**	**3.589**

Table 2: MDD-related statistics of some indices and funds available through the International Advisory Services Group, [1]. DCM=Diamond Capital Management; NLT=Non-Linear Technologies; OIC=Olsen Investment Corporation; TGF=Tradewinds Global Fund. The normalized Calmar ratio, $\overline{Calmar}$ is normalized to $\tau = 1$ yr. The relative strength index β is computed with respect to the $S\&P500$ as benchmark.

position. Specifically, the Calmar ratio of Π_1 is already normalized, i.e., $\overline{\text{Calmar}}_1 = 5$. If we compute the normalizing factors for portfolios Π_2 & Π_3, we get $\gamma(\Pi_2) = 0.74$ and $\gamma(\Pi_3) = 0.60$, from which we get the normalized Calmar ratios: $\overline{\text{Calmar}}_2 = 4.41$ and $\overline{\text{Calmar}}_3 = 3.62$. It is now clear that $\Pi_1 > \Pi_2 > \Pi_3$, if we normalize to $\tau = 1$. ■

The normalized Calmar ratio may depend on the choice of τ, the normalizing time. We can remove the τ-dependence by defining the *relative strength* $\beta(\Pi_1|\Pi_2)$ of portfolio Π_1 with respect to some other benchmark portfolio, Π_2. Π_2 could be (for example) the *S&P 500*. For normalizing time τ, define the *τ-relative strength* $\beta_\tau(\Pi_1|\Pi_2)$ of Π_1 with respect to Π_2,

$$\beta_\tau(\Pi_1|\Pi_2) = \frac{\overline{Calmar}_1(\tau)}{\overline{Calmar}_2(\tau)}.$$

If $\mathsf{Shrp}_1 \neq \mathsf{Shrp}_2$, then the τ-relative strength depends on τ. The limiting (i.e. $\tau \to \infty$) long term behavior of the relative strength is well defined, and so we define the *relative strength* $\beta(\Pi_1|\Pi_2) = \lim_{\tau\to\infty} \beta_\tau(\Pi_1|\Pi_2)$. One can show that

$$\textit{relative strength} = \beta(\Pi_1|\Pi_2) = \frac{Calmar_1}{Calmar_2} \times \frac{\frac{1}{T_1} Q_p(\frac{T_1}{2}\mathsf{Shrp}_1^2)}{\frac{1}{T_2} Q_p(\frac{T_2}{2}\mathsf{Shrp}_2^2)},$$

which is independent of τ. If the relative strength is greater equal to 1, then Π_1 is "better" than Π_2, written $\Pi_1 \succeq \Pi_2$. Since $\beta(\Pi_1|\Pi_3) = \beta(\Pi_1|\Pi_2)\beta(\Pi_2|\Pi_3)$, the relative strength index is *transitive* ($\Pi_1 \succeq \Pi_2$ and $\Pi_2 \succeq \Pi_3$ implies $\Pi_1 \succeq \Pi_3$), which is certainly a desirable consistency condition for any such strength index. It is *complete* and *anti-symmetric*, because $\beta(\Pi_1|\Pi_2) = 1/\beta(\Pi_2|\Pi_1)$ (so either $\Pi_1 \succeq \Pi_2$ or $\Pi_2 \succeq \Pi_1$ and $\Pi_1 \succeq \Pi_2 \implies \Pi_2 \preceq \Pi_2$). Thus $\succeq$ is a *total order*. Further, the choice of the reference instrument does not affect the total ordering, because $\beta(\Pi_1|\Pi_2) = \beta(\Pi_1|\Pi_3)/\beta(\Pi_2|\Pi_3)$ (so $\beta(\Pi_1|\Pi_3) \geq \beta(\Pi_2|\Pi_3) \implies \Pi_1 \succeq \Pi_2$). The relative strengths of the portfolios in the example, with Π_1 as benchmark, are given in Table 1.

Real Data. In Table 2, we give the MDD-related statistics for some indices and funds. The data (in non-bolded font) was obtained from the International Advisory Services Group, [1]. Notice that the expected MDD is generally slightly lower than predicted. One reason for this is due to the discretization bias (the data is built from monthly statistics, however the model is continuous). Notice that the time periods over which the funds are quoted are quite different, since the funds have been in existence for different periods of time. Some have not been around for 3 years, and some have been around significantly longer. Thus, it is not clear how to compare the funds using Calmar ratios for some standardized time period, 3 years being the norm in the industry: if a fund has been around less than 3 years, then it is not possible, and choosing (say) the most recent 3 year period for a well established fund ignores valuable data. However, the normalized Calmar ratios and the relative strengths facilitate seemless comparison among the funds using all the available data.

Summary. We now have a systematic way to quote Calmar ratios so that systems can be easily compared. Further, there is a direct (monotonic) relationship between the Calmar Ratio and the Sharpe Ratio. A deviation observed from historical data indicates a non-Brownian phenomenon at work, which could for example be the presence or absence of excessive correlation between successive loss periods, or the presense or absense of fat-tailed behavior for the returns (note however that it has been empirically found that higher moments have negligible impact on the Calmar ratio [4]). Such features may depend on the nature trading system, the types of markets (for example trending or mean reverting), and the degree of diversification. For example, for a passive buy and hold strategy, if the Calmar Ratio is lower than indicated by the theory, that could be due to positive autocorrelation for the returns, indicating the need for more risk control measures such as diversification or hedging. Alternatively, if a trend following system were to pick the trends accurately, then it could significantly improve the Calmar ratio.

Portfolio Aspects of MDD.

Mean variance analysis exploits the correlation structure between assets to build a portfolio with good Sharpe ratio characteristics. This ability is facilitated by the fact that the variance and return of a portfolio can be computed given these properties of the individual assets. As we have shown in the previous results, these parameters are also sufficient to obtain the $E(MDD)$ of the resulting portfolio, hence we should be able to perform such a similar analysis to optimize the MDD. Further, since the Calmar ratio is monotonic in the Sharpe ratio, we can directly transfer portfolio optimization methods for the Sharpe ratio over to the Calmar ratio. We briefly illustrate some of these issues here. Assume throughout that Calmar ratios are normalized to 1 year.

The Impact of Correlation. Consider for simplicity a portfolio of two instruments. If the correlation of the returns of the two instruments is low, then we should be able to construct a superior portfolio than either asset, from the risk-adjusted-return point of view. We want to quantify this effect using the previous analysis, and the Calmar Ratio as a performance

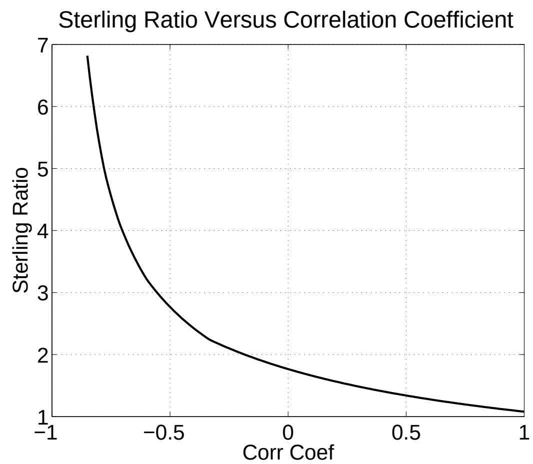

Figure 3: The Calmar Ratio for a Portfolio of Equally-Weighted Two Trading Systems/Markets (with $\mu_1 = \mu_2 = 0.2$, and $\sigma_1 = \sigma_2 = 0.2$) against the Correlation Coefficient of the Two Systems.

measure. For illustration, consider a portfolio in which the mean return of each instrument is 20%, and the standard deviation of the returns of each instrument is 20% (all annualized). Assume the portfolio is equal-weighted. Figure 3 shows the Calmar Ratio as a function of the correlation coefficient of the returns of the two instruments. While the fact that the Calmar Ratio decreases with increasing correlation is not surprising, the extent of the change is higher than expected. We should mention, however that it is quite difficult to find trading systems/markets both with positive returns and highly negative correlation. Highly negative correlations are typically achieved by a long-type system versus a short-type system, in which case their mean returns would typically be of opposite signs. So the part of the curve deep into the negative correlation portion is probably difficult to attain.

Can A losing System be Beneficial? It should, however, be possible to explore the negative correlation region by combining a losing system with a profitable system. To illustrate, let us perform the following curious experiment: consider two instruments with annualized returns $\mu_1 = \mu_2 = 20\%$, and standard deviations $\sigma_1 = \sigma_2 = 20\%$, with correlation coefficient $\rho_{12} = 0.8$ for the returns of the two instruments. Applying the formulas presented earlier, we find that the *best* Calmar ratio that can be achieved is 1.154, using a portfolio that weights each instrument equally. Assume now that we add to the portfolio a losing instrument with $\mu_3 = -10\%$, $\sigma_3 = 30\%$, and with negative correlations $\rho_{13} = \rho_{23} = -0.8$. Let the new weightings for the three instruments in the portfolio be 45%, 45%, and 10%. The Calmar Ratio for the augmented portfolio is now 1.308. This unexpected result shows how a *losing* trading system, that might initially be regarded as useless, is actually beneficial and leads to improved performance. The benefit of the negative correlation outweighs its lack of profit performance. It is as if this trading system/instrument provides “cohesion” to the portfolio. This instrument could for example be a shorted group of stocks or indices, thus providing the negative correlation with the rest of the portfolio of long stock positions. This result sheds some light into long-short portfolios. Not only do they serve as diversification vehicles by producing returns over different cycles than traditional long only portfolios, but they can produce better risk adjusted returns.

Even though correlation is currently considered in the industry as an important factor when deciding whether to add a trading system/instrument to a portfolio, it is usually second to the average return. With respect to risk adjusted returns, the correlation is almost on par with average returns, and deserves to be given a higher weight (when evaluating a trading strategy).

Conclusion.

The MDD is one of the most important risk measures. To be able to use it more effectively, its analytical properties have to be understood. As a step towards this direction, we have presented a review of some analytic results that we have developed as well as some applications of the analysis. In particular, we highlight the introduction of the normalized Calmar ratio as a way to compare quantitatively the Calmar ratios of portfolios over different time horizons. We also indicate the possibly underrated role of correlations in affecting the performance of portfolios, and these correlations can be systematically incorporated toward optimizing the Calmar ratio of a portfolio. We hope this study would spur more research analyzing this important risk measure.

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