

# The Equivalent Martingale Measure: An Introduction to Pricing Using Expectations

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## Abstract

We provide a self contained introduction to the *risk neutral* or *martingale* approach to the pricing of financial derivatives, while assuming no financial background. This approach to pricing provides a rich source of problems ideally suited to the application of Monte Carlo methods, thus forming a bridge between computational finance and some of the well developed tools available to engineers and scientists. We illustrate the power of the martingale approach by using it to develop the price of the European call option using only elementary methods and briefly discuss the pricing of the American put option as well as interest rate derivatives.

**Keywords:** Financial Derivative, Monte Carlo, Risk Neutral, Martingale, Pricing.

## 1 Introduction

Since Black and Scholes published their seminal paper, [2], the pricing of financial derivatives has been an active (and perhaps lucrative) area of research. A financial derivative is a financial instrument whose value (at some time  $t = T$  in the future) is completely determined by the price (or price history) of some other set of instruments (called the underlying instruments). The general goal is to price the derivative (obtain its fair value) at time  $t = 0$ . An abundance of financial derivatives exists in the market, and the interested reader is referred to the many excellent descriptions for a background (see for example [18]). The detailed nature of these financial derivatives will not play a central role in this paper and hence the method to be described is quite generally applicable.

The general approach to pricing is to first assume a certain (stochastic) model for the dynamics of the market. One would then like to know the fair value of a given financial instrument/derivative given the current state for the market, and the (assumed) stochastic model.

It is fair to assume that the possible future values of the instrument (and their associated probabilities) should play a central role in determining its present fair value.

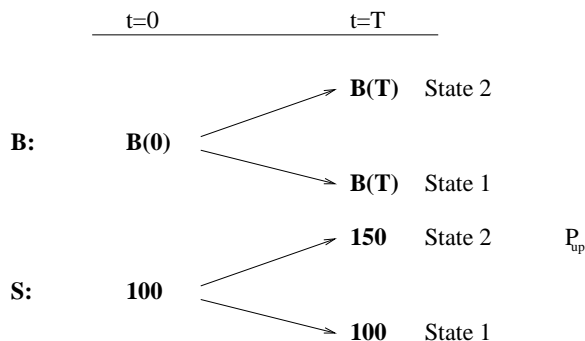
Generally, the fair price obtained for an instrument will be quite sensitive to the model that is chosen to describe the market dynamics. In the past, analytic solutions have generally been the rule, hence, only models that lend easily to tractable solutions could be used. These models are usually accompanied by unrealistic assumptions. The overwhelming increase in CPU speeds allows one to simulate market dynamics for quite complicated models, thus, efficient and accurate numerical methods provide an attractive alternative approach. Exactly how one would obtain the price by simulating the market is the subject of this presentation. It will become clear that advanced Monte Carlo techniques for obtaining the expected value of a stochastic variable will be the key tool.

The purpose of this paper is to introduce the ideas, and so we will restrict our attention to finite state economies. While the step to infinite state economies is conceptually small, certain technical difficulties present themselves and the reader is referred to more advanced discussions on asset pricing (see for example [6]). For the rich history of this field, the reader is referred to the excellent discussions in [18], [27] and [30]. More mathematical treatments can be found in [31], [23], [19] and [6]. Further reading on Monte Carlo approaches can be found in [3] and [28], and martingale methods are discussed extensively in [22] and [16].

The outline of the paper follows. To get the ball rolling, we begin with an example, which will lead us to the more formal discussion of the martingale measure. In section 4 we consider several examples, including the European call option, the American put option and an elementary interest rate derivative.

## 2 Example – Two Period Economy

Consider the following two period economy. At  $t = 0$ , two instruments are available. Any amount (even fractional) of either instrument may be sold or purchased at the specified market price - i.e., arbitrary short or long positions are allowed. A *risk free* asset or *bond*,  $B$ , and a *stock*,  $S$ . At  $t = 0$  (the first period), the bond is worth  $B(0)$ , and, the stock is worth  $S(0) = 100$ . At  $t = T$  (the second period), the economy can be in one of two states. In both states, the bond is valued at  $B(T)$  and hence is risk free. In the first state the stock is valued at  $S(T) = 100$  and in the second state, the stock is valued at  $S(T) = 150$ .

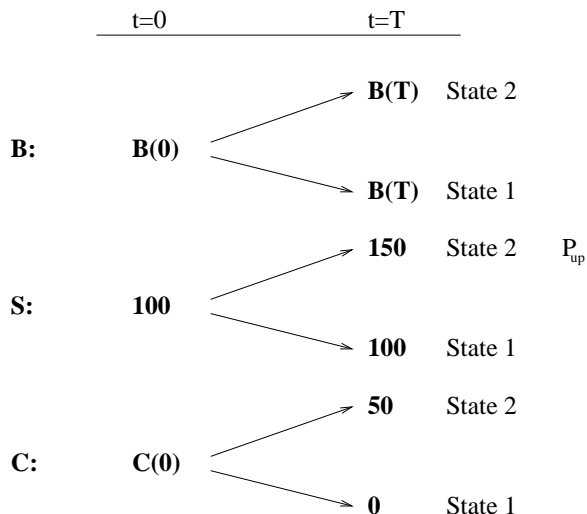


To complete our model for the market, we need to specify the probabilities that the market will be in states 1 and 2 respectively. Suppose that  $P_{up}$  is the probability that the market goes

up. We have thus completely specified the market dynamics for our simple economy. In this simplified economy, it is clear that one can guarantee an amount  $B(T)$  at  $t = T$  by investing  $B(0)$  in the bond at  $t = 0$ . The *risk free discount factor* is defined by

$$D(T) = \frac{B(0)}{B(T)} = e^{-rT} \quad (1)$$

where  $r > 0$  is the risk free interest rate for continuous compounding. The economy becomes more interesting when we introduce a third instrument whose values at  $t = T$  in the possible states of the economy are known. Any instrument will do, so, for illustration we will use the *European call option*<sup>1</sup> on the stock with a strike price of 100, expiring at time  $T$ . At  $t = T$ , the call option will either be worth 50, if the market went up, or 0 if not. The complete picture of the market is now summarized in the following figure



The question we would like to answer is “What should the price of the call option be at  $t = 0$ , i.e., what is  $C(0)$ ?”. Suppose that  $P_{up}$  is very close to 1. Holding the option will “almost” guarantee 50 at  $t = T$ . hence one might expect that<sup>2</sup>  $C(0) \rightarrow D(T) \times 50$  as  $P_{up} \rightarrow 1$ . On the other hand, if  $P_{up}$  is very close to 0, then  $C(T) = 0$  with very high probability and we should not be willing to pay anything for it at  $t = 0$ . Hence, one might expect  $C(0) \rightarrow 0$  as  $P_{up} \rightarrow 0$ . In general, we might expect that  $C(0)$  will be some function of  $P_{up}$ . A good guess would be that the value at  $t = 0$  should be the expectation of the value of the instrument at  $t = T$ , discounted to  $t = 0$  by multiplying by  $D(T)$ . In doing this we are treating the call option as if it were a riskless asset, yielding its expected value at  $t = T$ . Since the option is actually risky, in general we might expect that we should not have to pay as much (since we are absorbing this risk), hence, we generally expect that

$$C(0) \leq D(T) \times E[C(T)] \quad (2)$$

<sup>1</sup>A European call option with strike  $K$  expiring at time  $T$  gives the holder the option to purchase one unit of stock at the strike price  $K$  at time  $T$ .

<sup>2</sup>we multiply by  $D(T)$  to be consistent with the bond dynamics, namely, a sure  $B(T)$  at time  $t = T$  is worth  $B(0)$  at  $t = 0$ , hence, a sure 50 at time  $t = T$  should be worth  $B(0) \times 50/B(T)$  at  $t = 0$ .

The difference between the left hand side and the right hand side is sometimes denoted the *risk premium*.

The surprising thing, as we are about to see, is that  $C(0)$ , the price to be paid for the option, is independent of  $P_{up}$ , provided that  $0 < P_{up} < 1$ . All the (apparently) intuitive arguments made in the last paragraph (taking  $P_{up}$  close to 1 or 0) are actually flawed. In fact, even (2) need not necessarily hold. To understand why this is so, we need to make use of the intuitive notion that it should not be possible to guarantee a non-negative profit in every possible future state of the market, and a positive profit in some states, using a zero initial investment. Informally, we then have the following *mini*-theorem. The proof is not essential for the moment, but is presented below for completeness.

**mini-theorem** If  $0 < P_{up} < 1$ , and, one cannot make money out of nothing (or less), then

1.  $2/3 < D(T) < 1$
2.  $C(0) = 100(1 - D(T))$

independent of  $P_{up}$ .

PROOF: The first claim is proved as follows. Suppose that  $D(T) \leq 2/3$ . Then, at  $t = 0$ , sell one unit of stock and buy 100 dollars worth of bond. The net investment is zero<sup>3</sup>. At  $t = T$  the bond is worth at least 150 (as  $D(T) \leq 2/3$ ) and since you owe one unit of stock, the profit in state 1 is at least 50 and in state 2 is at least 0. Thus, with no investment, you can guarantee a non-negative profit in all states, and a positive profit in some states. This is not possible, hence  $D(T) > 2/3$ . Similarly, one shows that  $D(T) < 1$  by buying one unit of stock and selling 100 dollars worth of bond. To prove the second claim, suppose that  $C(0) = 100(1 - D(T))(1 + \rho)$ , where  $\rho > 0$ . At  $t = 0$ , buy 1 unit of stock, and finance this by selling  $100/B(T)$  units of bond and selling  $1/(1 + \rho)$  units of call option. The net investment is zero. At  $t = T$ , if state 1 occurs, then the net profit is zero, but if state 2 occurs, the net profit is  $50\rho/(1 + \rho) > 0$ . Once again, this should not be possible, therefore  $\rho \leq 0$ . To show that  $\rho < 0$  is not allowed, we repeat the same argument but instead we sell the stock and buy the bond and option. A zero net investment then yields a non-negative profit in every state and a positive profit in some states. Thus  $\rho = 0$ , concluding the proof. ■

The important things to note are:

1. The exact nature of the instruments that constitute the market are not relevant, only that they follow the specified market dynamics.
2. To price the third instrument (given  $D(T)$ , or alternatively, the interest rate), it was not necessary to know  $P_{up}$  (the probabilities of the various states). One only needs to know what states are *possible*. Among other things, this implies the counter intuitive fact that the price of the option would be the same whether  $P_{up} = 1 - 10^{-100}$  or  $P_{up} = 10^{-100}$ .

That the actual probability  $P_{up}$  plays no role in the pricing of the instruments is not a quirk of the particular economy that we happen to have chosen. In fact, in the next section, we give the generalization of our mini-theorem to an arbitrary finite state, finite instrument, two period economy. Our approach here was to “guess by some magic” that  $C(0) = 100(1 - D(T))$ , and

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<sup>3</sup>We ignore transaction costs here and throughout.

then we proved that it must be so. A purely mechanical and systematic approach to getting  $C(0)$  will unfold in the next section. In order to proceed, we need to formalize some of the notions described already, such as the phrase “not being able to make money out of nothing or less”.

### 3 The Risk-Neutral/Martingale Measure

First, we need to set up the notation to describe the economy/market. Suppose that there are  $N$  instruments, whose prices at time  $t$  are given by  $S_i(t)$ ,  $i = 1, \dots, N$  which we will denote by the column vector  $\mathbf{S}(t)$ , that represents the state of the economy at time  $t$ . For the moment, we focus on the two times  $t = 0$  and  $t = T$ . Suppose that at  $t = T$ , there are  $K$  possible states of the world, in other words, the state  $\mathbf{S}(T)$  can be one of the  $K$  possible vectors  $\mathbf{S}(T)^{(1)}, \dots, \mathbf{S}(T)^{(K)}$ . Let the probability of state  $j$  occurring at  $t = T$  be  $P_j$ ,  $j = 1 \dots K$ , represented by the column vector  $\mathbf{P}$ . We define the payoff matrix at  $t = T$  as the matrix  $\mathbf{Z}(T) = [\mathbf{S}(T)^{(1)}, \mathbf{S}(T)^{(2)}, \dots, \mathbf{S}(T)^{(K)}]$ . The element  $\mathbf{Z}_{ij}$  represents the value of one unit of instrument  $i$  in state  $j$  for  $i = 1, \dots, N$  and  $j = 1, \dots, K$ . A particular row of  $\mathbf{Z}$  represents the possible values that a particular instrument can take at time  $T$ . An instrument is risk free if its value in every state is a constant (i.e., if the row is a constant). A particular column represents the values of all the instruments in a particular state. Having specified  $\mathbf{S}(t)$ ,  $\mathbf{Z}(T)$  and  $\mathbf{P}$ , we have fully specified our economy, including its dynamics. For the example in the previous section, we have

$$\mathbf{S}(0) = \begin{bmatrix} B(0) \\ 100 \\ C(0) \end{bmatrix} \quad \mathbf{Z}(T) = \begin{bmatrix} B(T) & B(T) \\ 100 & 150 \\ 0 & 50 \end{bmatrix} \quad \mathbf{P}(T) = \begin{bmatrix} 1 - P_{up} \\ P_{up} \end{bmatrix} \quad (3)$$

A portfolio  $\Theta$  is a column vector of  $N$  components that indicates how many units of each instrument is held. The value of such a portfolio at time  $t$  is given by  $\Theta^T \mathbf{S}(t)$ . We use the notation  $(\cdot)^T$  to denote the transpose. Thus, the value of the portfolio at  $t = 0$  is  $\Theta^T \mathbf{S}(0)$ , the possible values that the portfolio can take on at time  $T$  are given by the row vector  $\Theta^T \mathbf{Z}(T)$ , and the expected value of the portfolio at time  $T$  is given by  $\Theta^T \mathbf{Z}(T) \mathbf{P}(T)$ .

Using this notation, let's re-examine the proof of the mini-theorem in the previous section. Consider the case that  $C(0) = 100(1 - D(T))(1 + \rho)$  where  $\rho > 0$ . Construct the portfolio

$$\Theta = \begin{bmatrix} -100/B(T) \\ 1 \\ -1/(1 + \rho) \end{bmatrix} \quad (4)$$

At  $t = 0$ , the value of this portfolio is

$$\begin{bmatrix} \frac{-100}{B(T)} & 1 & \frac{-1}{(1 + \rho)} \end{bmatrix} \begin{bmatrix} B(0) \\ 100 \\ 100(1 - D(T))(1 + \rho) \end{bmatrix} = 0 \quad (5)$$

At  $t = T$ , the possible values of the portfolio are<sup>4</sup>

$$\begin{bmatrix} \frac{-100}{B(T)} & 1 & \frac{-1}{(1+\rho)} \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ 100 & 150 \\ 0 & 50 \end{bmatrix} = \begin{bmatrix} 0 & \frac{50\rho}{1+\rho} \end{bmatrix} \geq \mathbf{0} \quad (6)$$

We want to rule out these kinds of portfolios because any “sane” individual would want to hold an unlimited amount of such portfolios, as they can only result in a gain (with no initial investment). This would lead to disequilibrium in the market, hence, we are led to the following definitions.

**Definition 3.1 (Type I Arbitrage)** *An arbitrage opportunity of type I exists if and only if there exists a portfolio  $\Theta$  such that*

$$\Theta^T \mathbf{S}(0) \leq 0 \text{ and } \Theta^T \mathbf{Z}(T) \geq 0 \quad (7)$$

**Definition 3.2 (Type II Arbitrage)** *An arbitrage opportunity of type II exists if and only if there exists a portfolio  $\Theta$  such that*

$$\Theta^T \mathbf{S}(0) < 0 \text{ and } \Theta^T \mathbf{Z}(T) \geq 0 \quad (8)$$

An economy in equilibrium cannot sustain either type I or type II arbitrage opportunities. In words, an investment that guarantees a future non-negative return, with a possible positive return, must have a positive cost today, and, an investment that guarantees a non-negative return in the future must cost a non-negative amount today. Note that these two types of arbitrage opportunities are distinct in that the existence of type I arbitrage does not in general imply the existence of type II, and the existence of type II in general does not imply the existence of type I. If there exists a risk free asset with positive value at  $t = 0$  and  $t = T$ , then by adding a suitable amount of the risk free asset to the portfolio one can convert type II arbitrage to type I arbitrage.

We are now ready for the main theorem. The statement of the theorem may appear a little abstract, but as we shall soon see, it essentially tells us that we can price an instrument at  $t = 0$  (i.e., today) by taking an expectation over its future values at  $t = T$  (i.e., tomorrow).

**Theorem 3.3 (Positive supporting price)** *The following statements are equivalent.*

1. *There do not exist arbitrage opportunities of type I or type II.*
2. *There exists  $\psi > 0$  such that  $\mathbf{S}(0) = \mathbf{Z}(T)\psi$ , or more explicitly,  $\exists \psi > 0$  such that*

$$\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1K} \\ Z_{21} & Z_{22} & \dots & Z_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & \dots & Z_{NK} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_K \end{bmatrix} \quad \psi_i > 0 \quad (9)$$

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<sup>4</sup>For vectors, we use the notation  $\mathbf{x} \leq 0$  to signify a vector that has every component  $\leq 0$  and at least one component  $< 0$ .  $\mathbf{x} \geq 0$  if and only if  $-\mathbf{x} \leq 0$ . We use the notation  $\mathbf{x} \leq 0$  to signify a vector with every component  $\leq 0$ , allowing the possibility that every component  $= 0$ .  $\mathbf{x} \geq 0$  if and only if  $-\mathbf{x} \leq 0$ .

where we have the understanding that  $S$  refers to  $t = 0$  and  $Z$  refers to  $t = T$ . Since this is the central theorem, we provide an elementary proof based upon some fundamental concepts from the theory of linear programming in the appendix in section A. The essential content of the theorem is that if there is no arbitrage, then a positive price vector  $\psi$ , exists such that the prices today are given by a weighted sum of the prices tomorrow, where the weightings are given by the price vector  $\psi$ , and, are the *same* for each instrument. This looks suspiciously like an expectation over the possible prices tomorrow taken over the distribution  $\psi$ . To fully appreciate this, we need to massage the theorem into a more suitable form. First, notice that only the relative prices should matter. To this end, we pick an instrument with respect to which we price all other instruments. This instrument is called the *numeraire*, which for our purposes is just a fancy word for reference instrument. Often, one chooses the numeraire to be the risk free instrument if one exists<sup>5</sup>. Without loss of generality, assume that the first instrument is chosen as numeraire. Rewriting (9), we have,

$$\begin{bmatrix} 1 \\ S_2/S_1 \\ \vdots \\ S_N/S_1 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 \\ Z_{21}/Z_{11} \\ \vdots \\ Z_{N1}/Z_{11} \end{pmatrix} & \begin{pmatrix} 1 \\ Z_{22}/Z_{12} \\ \vdots \\ Z_{N2}/Z_{12} \end{pmatrix} & \cdots & \begin{pmatrix} 1 \\ Z_{2K}/Z_{1K} \\ \vdots \\ Z_{NK}/Z_{1K} \end{pmatrix} \end{bmatrix} \begin{bmatrix} \psi_1 Z_{11}/S_1 \\ \psi_2 Z_{12}/S_1 \\ \vdots \\ \psi_K Z_{1K}/S_1 \end{bmatrix} \quad (10)$$

Notice that for each instrument, we are comparing its price in a given state to the price of the numeraire in that state. Define a vector  $\tilde{\mathbf{P}}$  by

$$\tilde{P}_i = \frac{Z_{1i}}{S_1} \psi_i \quad (11)$$

Then, assuming that all prices are positive and since  $\psi_i > 0$ , we see that  $\tilde{\mathbf{P}} > 0$ . Further, from the first row in (10),  $\sum_i \tilde{P}_i = 1$ , thus  $\tilde{\mathbf{P}}$  is a probability vector. Further, the remaining rows in (10) give that

$$\frac{S_i}{S_1} = \sum_{j=1}^K \tilde{P}_j \frac{Z_{ij}}{Z_{1j}} \quad (12)$$

We have now arrived at the key pricing theorem.

**Theorem 3.4 (Equivalent Martingale Measure)** *There do not exist arbitrage opportunities of type I or II if and only if there exists a probability vector  $\tilde{\mathbf{P}}$ , called an equivalent martingale measure such that*

$$\frac{S_i(0)}{S_1(0)} = E_{\tilde{\mathbf{P}}} \left[ \frac{S_i(T)}{S_1(T)} \right] \quad (13)$$

*In other words,  $S_i(T)/S_1(T)$  is a martingale under the measure  $\tilde{\mathbf{P}}$ .*

The measure  $\tilde{\mathbf{P}}$  is also often referred to as the *risk-neutral measure*, or the *risk-adjusted probabilities*. If there is a risk free asset, one usually chooses this as the numeraire, in which case

$$S_i(0) = E_{\tilde{\mathbf{P}}} \left[ \frac{S_1(0)}{S_1(T)} S_i(T) \right] = E_{\tilde{\mathbf{P}}} [D(T) S_i(T)] = D(T) E_{\tilde{\mathbf{P}}} [S_i(T)] \quad (14)$$

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<sup>5</sup>It should be clear that there cannot exist two risk free instruments offering different returns.

where  $D(T)$  factors out of the expectation because the numeraire is a risk free asset and thus is constant in every state. If the numeraire is not a risk free asset, then this step is not valid and the “discount factor” must remain inside the expectation. Thus, *the price of an instrument today is given by the discounted expectation of its future price, where the expectation is with respect to the risk neutral measure.* To illustrate how this theorem is applied, we continue with the example that was presented earlier. By theorem 3.4, there is a probability vector  $\tilde{\mathbf{P}} > 0$  such that

$$\begin{bmatrix} 1 \\ 100/B(0) \\ C(0)/B(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 100/B(T) & 150/B(T) \\ 0 & 50/B(T) \end{bmatrix} \begin{bmatrix} \tilde{P}_{down} \\ \tilde{P}_{up} \end{bmatrix} \quad (15)$$

The first two rows yield that  $\tilde{P}_{up} = 2(1/D(T) - 1)$  and  $\tilde{P}_{down} = 1 - \tilde{P}_{up}$ , hence, because  $0 < \tilde{P}_{up} < 1$ , we see that  $2/3 < D(T) < 1$ . The third row then yields  $C(0) = 100(1 - D(T))$ . We have thus (mechanically) reproduced the mini-theorem of the previous section with relative ease.

In general, if the number of states in the economy is large, the risk neutral measure will not be unique. More generally, suppose that the rank of  $\mathbf{Z}$  is  $k$ , which represents the number of independent instruments. Then, we see that if the number of states ( $K$ ) in the next period is less than or equal to  $k$ , we can choose  $K$  linearly independent rows of  $\mathbf{Z}$  to solve uniquely for the risk neutral probabilities (by matrix inversion). If all of these risk neutral probabilities are not real probabilities, then an arbitrage opportunity exists. If the risk neutral probabilities exist, then they can be used to price *all* the instruments. If, on the other hand,  $k < K$ , then there are either infinitely many possible risk neutral probabilities (in which case, there are many pricings that are consistent with no arbitrage) or there are none (in which case, there is an arbitrage opportunity). As a further note, the the risk neutral probabilities can be obtained without paying any attention to the actual probabilities with which the future states occur. One simply needs to agree on what states are *possible*.

## Change of Numeraire

In an economy where the risk neutral probabilities are well defined, they still need not be unique. It is possible for different people to disagree as to the choice of numeraire. The choice of numeraire is arbitrary, and a particular numeraire is chosen in order to make subsequent calculations easier (for example it may be easier to derive the risk neutral measure for a particular numeraire). However, independent of the choice of numeraire, provided that the correct expectation is computed, the prices obtained for the assets at  $t = 0$  are unique. It is possible that the risk neutral measure is more easily obtained with a particular numeraire, but that the expectations are more conveniently computed with respect to another numeraire. If this is the case, then one needs to change the risk neutral probabilities to the relevant numeraire. Suppose that with numeraire  $S_i$  one has computed the martingale measure  $\tilde{\mathbf{P}}$ , but that one wishes to compute the expectations with  $S_j$  as numeraire. By direct substitution into (13) it can be verified that the new martingale measure  $\tilde{\mathbf{P}}'$  that reflects this *change of numeraire* is given by

$$\tilde{P}'_{\alpha} = \frac{S_i/Z_{i\alpha}}{S_j/Z_{j\alpha}} \tilde{P}_{\alpha} \quad (16)$$



In particular, all one needs to know are the discount factors for the two instruments in the various future states. Such issues are discussed in more detail in [14].

### 3.1 Continuous State Economies

We have restricted our attention to a finite state economy. We will here present a very brief treatment of the continuous state (but finite instrument) economy. More details can be found in some of the references.

The future state can be indexed by a random variable  $\tilde{s}$  which we assume to have a probability density function  $\pi(s)$ . Then the price matrix at  $t = T$  becomes  $\tilde{\mathbf{Z}}(\tilde{s})$ , a random vector. Given a portfolio  $\Theta$ , its value at  $t = T$  is a random variable  $\tilde{V}_\Theta(s) = \Theta^\top \tilde{\mathbf{Z}}(s)$ . Then, type I arbitrage would require the existence of a portfolio that satisfies

$$\Theta^\top \mathbf{S}(0) \leq 0, \quad \tilde{V}_\Theta \geq 0, \quad P[\tilde{V}_\Theta > 0] > 0 \quad (17)$$

and type II arbitrage requires the existence of a portfolio that satisfies

$$\Theta^\top \mathbf{S}(0) < 0, \quad \tilde{V}_\Theta \geq 0 \quad (18)$$

The analog of theorem 3.3 is that under suitable regularity conditions, the absence of arbitrage opportunities is equivalent to the existence of a positive integrable function  $\psi(s)$  such that,

$$\mathbf{S}(0) = \int ds \tilde{\mathbf{Z}}(s) \psi(s), \quad \psi(s) > 0 \quad (19)$$

Further, choosing a numeraire (instrument 1) and redefining  $\tilde{\pi}(s) = \psi(s) \tilde{\mathbf{Z}}_1(s) / S_1(0)$ , we have that

$$S_i(0) = \int \frac{S_1(0)}{\tilde{\mathbf{Z}}_1(s)} \tilde{\mathbf{Z}}_i(s) \tilde{\pi}(s) = E_{\tilde{\pi}}[D(T)(s) \tilde{\mathbf{Z}}_i] \quad (20)$$

## 4 More Examples

The previous section suggests an approach to pricing which we summarize here.

Let  $K$  be the number of possible future states for the economy, and let the number of independent instruments be  $N = K' + n$ . Without loss of generality, we assume that the dynamics of the first  $K'$  (underlying) instruments are completely specified, including their current prices. The remaining  $n$  instruments are derivatives of the first  $K'$  instruments, in that their values in the possible future states are known, given the values of the underlying instruments. It is necessary to obtain the correct prices of the  $n$  derivatives at  $t = 0$ .

1. Choose a numeraire (reference instrument), and call this  $S_1$ .
2. Solve the set of  $K'$  simultaneous linear equations in the  $K$  unknowns  $\tilde{P}_i > 0$ ,  $i = 1 \dots K$ ,

$$\frac{S_i}{S_1} = \sum_{j=1}^K \frac{Z_{ij}}{Z_{1j}} \tilde{P}_j \quad \text{for } i = 1, \dots, K' \quad (21)$$

to obtain the martingale measure  $\tilde{\mathbf{P}}$ .

3. Obtain the risk neutral prices of all the instruments as follows

$$S_i(0) = S_1(0) E_{\tilde{\mathbf{P}}} \left[ \frac{S_i(T)}{S_1(T)} \right] = E_{\tilde{\mathbf{P}}} [D(T)S_i(T)] \stackrel{(a)}{=} D(T)E_{\tilde{\mathbf{P}}} [S_i(T)] \quad (22)$$

where  $D(T) = S_1(0)/S_1(T)$  and (a) only follows if  $S_1$  is a risk free instrument.

4. These prices (if the measure exists) are unique if  $K' > K$ . If such a measure cannot be found, then there exist arbitrage opportunities.

The connection with Monte Carlo techniques should now be clear. It is often possible to obtain the risk neutral measure, but due to the complexity of derivatives, it is often not possible to compute the desired expectation analytically. One still needs to price the derivatives efficiently, and an ideal tool for obtaining the necessary expectation is a Monte Carlo simulation. The following examples will illustrate the technique and how Monte Carlo simulations can be useful.

## 4.1 Stock and Bond Economy

We would like to generalize our first example as follows. Suppose the economy contains a stock and a risk free asset whose value grows according to a compounding interest rate. This economy (and its dynamics) can be fully specified by giving  $\mathbf{S}(0)$ ,  $\mathbf{Z}(\Delta)$  and  $\mathbf{P}$ .

### 4.1.1 Two State Stock Dynamics - Binary Lattice Models

In this dynamics, the stock, at time  $t = \Delta$  can exist in an up state,  $S(0)\lambda_+$ , or a down state,  $S(0)\lambda_-$ .

$$\mathbf{S}(0) = \begin{bmatrix} B(0) \\ S(0) \end{bmatrix} \quad \mathbf{Z}(\Delta) = \begin{bmatrix} B(0)e^{r\Delta} & B(0)e^{r\Delta} \\ S(0)\lambda_+ & S(0)\lambda_- \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} P_u \\ 1 - P_u \end{bmatrix} \quad (23)$$

$$\lambda_{\pm} = 1 + \mu\Delta \pm \sigma\sqrt{\Delta}$$

The particular dependence on  $\Delta$  for the two possible values of  $\mathbf{S}(\Delta)$  are chosen with a view toward taking the limit  $\Delta \rightarrow 0$ . By summing up such changes for infinitesimal times, we can get the change for a finite time. In order to obtain acceptable behavior for finite times, one must choose this type of dependence for the infinitesimal changes. This will become clearer in the following discussion. Setting up and solving (10) for  $\tilde{P}_u$ , we have

$$\begin{bmatrix} 1 \\ \frac{S(0)}{B(0)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{\mathbf{S}(0)\lambda_+}{B(0)e^{r\Delta}} & \frac{\mathbf{S}(0)\lambda_-}{B(0)e^{r\Delta}} \end{bmatrix} \begin{bmatrix} \tilde{P}_u \\ 1 - \tilde{P}_u \end{bmatrix} \quad (24)$$

$$\Rightarrow \tilde{P}_u = \frac{e^{r\Delta} - \lambda_-}{\lambda_+ - \lambda_-} = \frac{1}{2} \left( 1 + \frac{(r - \mu)\sqrt{\Delta}}{\sigma} + O(\Delta) \right) \quad (25)$$

The limit  $\Delta \rightarrow 0$  will be useful later. If the real dynamics is represented by  $P_u = \frac{1}{2}$ , then the change in  $S$  can be represented by the following dynamics  $\Delta S = \mu S \Delta t + \sigma S \Delta W$ , where  $\Delta W$  is a random variable with mean zero and variance  $\Delta t$ . One can convince oneself that by

suitably changing  $\mu$ , the restriction to  $P_u = \frac{1}{2}$  loses no generality. In the risk neutral measure, however,

$$\begin{aligned} E_{\tilde{\mathbf{P}}}[\Delta S] &= S(\mu\Delta + \sigma\sqrt{\Delta})\tilde{P}_u + S(\mu\Delta - \sigma\sqrt{\Delta})(1 - \tilde{P}_u) \\ &= S(e^{r\Delta} - 1) = rS\Delta + O(\Delta^2) \end{aligned} \quad (26)$$

$$\begin{aligned} Var_{\tilde{\mathbf{P}}}[\Delta S] &= S^2 \left( \mu\Delta + \sigma\sqrt{\Delta} - (e^{r\Delta} - 1) \right)^2 \tilde{P}_u + S^2 \left( \mu\Delta - \sigma\sqrt{\Delta} - (e^{r\Delta} - 1) \right)^2 (1 - \tilde{P}_u) \\ &= \sigma^2 S^2 \Delta \left( 1 - \frac{(e^{r\Delta} - 1 - \mu\Delta)^2}{\sigma^2 \Delta} \right) = \sigma^2 S^2 \Delta + O(\Delta^2) \end{aligned} \quad (27)$$

Comparing means and variances, we see that the risk neutral dynamics can be represented by the following equation (to leading order in  $\Delta t$ )

$$\Delta S = rS\Delta t + \sigma S\Delta\tilde{W} \quad (28)$$

where again  $\Delta\tilde{W}$  is a random variable with mean zero and variance  $\Delta t$ .  $\mu$  does not appear in the dynamics, which could have been anticipated, because, as already mentioned, in the real world dynamics we could have absorbed  $\mu$  into  $P_u$ , the real-world probability of an up move. Since the risk neutral probabilities do not make reference to the real world probabilities at all,  $\mu$  should thus not appear in the risk neutral dynamics. In otherwords, we could have just as well started with the real world dynamics

$$\mathbf{S}(0) = \begin{bmatrix} B(0) \\ S(0) \end{bmatrix} \quad \mathbf{Z}(\Delta) = \begin{bmatrix} B(0)e^{r\Delta} & B(0)e^{r\Delta} \\ S(0)\lambda_+ & S(0)\lambda_- \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} P_u \\ 1 - P_u \end{bmatrix} \quad (29)$$

where  $\lambda_{\pm} = 1 \pm \sigma\sqrt{\Delta}$ , and the risk neutral dynamics would have been identical to (28) in the small  $\Delta$  limit.

#### 4.1.2 Three State Stock Dynamics

In this dynamics, the stock can exist in an “unchanged”, up or down state in the next time period.

$$\mathbf{S}(0) = \begin{bmatrix} B(0) \\ S(0) \end{bmatrix} \quad \mathbf{Z}(\Delta) = \begin{bmatrix} B(0)e^{r\Delta} & B(0)e^{r\Delta} & B(0)e^{r\Delta} \\ S(0)\lambda_+ & S(0)(1 + \mu\Delta) & S(0)\lambda_- \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} P_u \\ P_0 \\ P_d \end{bmatrix} \quad (30)$$

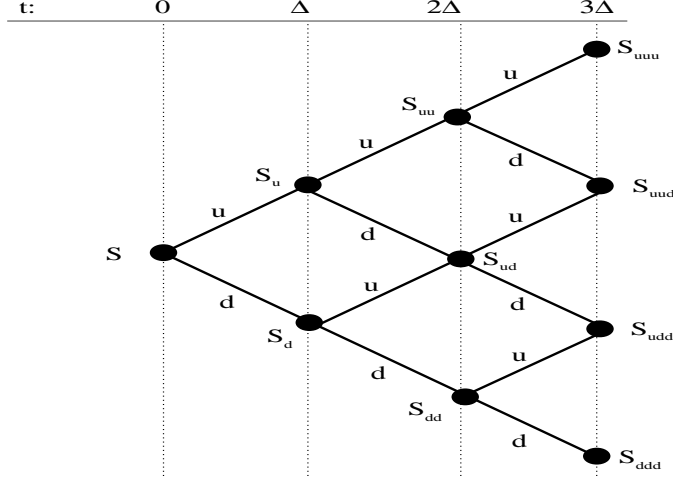
where  $\lambda_{\pm} = 1 + \mu\Delta \pm \sigma\sqrt{\Delta}$ . In this case, there are more states than there are instruments and the risk neutral measure is not unique. There are infinitely many risk neutral probabilities that would be consistent with no arbitrage and one would have to resort to some other means to pick one of these - for example through a utility framework.

## 4.2 Multi-period Binary Lattice and the European Call Option

In this example we show how to extend the one period binary lattice model already considered to the multi-period case and derive the price for the European call option without assuming

any knowledge of stochastic differential equations. Such binary lattices are often used to model economies (see for example [24]).

First we would like to argue that the risk neutral probabilities may be treated like ordinary probabilities. As such, they may be multiplied to obtain the probability of independent events. We illustrate with an example using the three period economy.



Applying the risk neutral pricing at  $t = 0$  we see that

$$S = e^{-r\Delta}(S_u\tilde{P}_u + S_d\tilde{P}_d) \quad (31)$$

Applying the risk neutral pricing at  $t = \Delta$  we see that  $S_u = e^{-r\Delta}(S_{uu}\tilde{P}_u + S_{ud}\tilde{P}_d)$  and  $S_d = e^{-r\Delta}(S_{ud}\tilde{P}_u + S_{dd}\tilde{P}_d)$  giving that

$$S = e^{-2r\Delta}(S_{uu}\tilde{P}_u^2 + 2S_{ud}\tilde{P}_u\tilde{P}_d + S_{dd}\tilde{P}_d^2) \quad (32)$$

Further, using  $S_{uu} = e^{-r\Delta}(S_{uuu}\tilde{P}_u + S_{uud}\tilde{P}_d)$ ,  $S_{ud} = e^{-r\Delta}(S_{uud}\tilde{P}_u + S_{udd}\tilde{P}_d)$  and  $S_{dd} = e^{-r\Delta}(S_{udd}\tilde{P}_u + S_{ddd}\tilde{P}_d)$ , we find that

$$S = e^{-3r\Delta}(S_{uuu}\tilde{P}_u^3 + 3S_{uud}\tilde{P}_u^2\tilde{P}_d + 3S_{udd}\tilde{P}_u\tilde{P}_d^2 + S_{ddd}\tilde{P}_d^3) \quad (33)$$

By inspecting this formula, we can pick out that  $\tilde{P}_{u^3} = \tilde{P}_u^3$ ,  $\tilde{P}_{u^2d} = 3\tilde{P}_u^2\tilde{P}_d$ ,  $\tilde{P}_{ud^2} = 3\tilde{P}_u\tilde{P}_d^2$  and  $\tilde{P}_{d^3} = \tilde{P}_d^3$ . It is apparent that the risk neutral probabilities at the  $N^{th}$  time period will be given by a binomial distribution of order  $N$

$$\tilde{\mathbf{P}} \left[ S(N) = S(0)u^k d^{N-k} \right] = \binom{N}{k} \tilde{P}_u^k (1 - \tilde{P}_u)^{N-k} \quad (34)$$

We do not prove this here, but a proof induction should be plausible at this point.

The *European call option* with expiry at time  $T$  and strike at  $K$  gives the holder the right (but not the obligation) to buy the stock at time  $t = T$  at the strike price  $K$ . Clearly the value of this option (and hence its price) at time  $T$  is given by  $(S(T) - K)\Theta(S(T) - K)$ , where  $\Theta(x) = 0$  for  $x < 0$  and  $\Theta(x) = 1$  otherwise. Suppose that we consider the time interval  $[0, T]$  broken into  $N$  smaller intervals of size  $\Delta = T/N$ . If  $\Delta$  is small enough, it is reasonable to

assume that the stock could only evolve into an up or down state during the time interval. In which case, the multi-period binary lattice dynamics would apply and the price of the call option at time  $t = 0$ ,  $C(0)$ , is given by taking the discounted expectation with respect to the risk neutral measure given in (34). Thus,

$$\begin{aligned} C(0) &= C(S(0), K, r, \sigma, \mu, T) = e^{-rT} E_{\tilde{\mathbf{P}}}[ (S(T) - K)\Theta(S(T) - K) ] \\ &= e^{-rT} \sum_{k=0}^N \binom{N}{k} \left( S(0)\lambda_+^k \lambda_-^{N-k} - K \right) \tilde{P}_u^k (1 - \tilde{P}_u)^{N-K} \Theta \left( S(0)\lambda_+^k \lambda_-^{N-k} - K \right) \end{aligned} \quad (35)$$

If we insist on using a discrete time representation of our market, then there is no option but to compute this expectation numerically. The natural approach would be to use Monte Carlo simulations for more complex dynamics. A key issue is the efficiency of the computation. Given  $\mu, \sigma$  and assuming that a proxy for the interest rate  $r$  exists, one only needs to compute a single expectation. Often  $\mu$  and  $\sigma$  are not known. In fact, one needs to *calibrate* the model to the observed option data to extract  $\mu$  and  $\sigma$ . This usually involves an optimization in the  $\mu$  and  $\sigma$  space, which is extremely computationally intensive.

A useful idealization is to take the limit  $N \rightarrow \infty$ . In fact, one of the reasons for choosing the particular dependence of  $\lambda_{\pm}$  on  $\Delta = T/N$  is so that this limit is physically plausible (any other dependence would have rendered this limit either trivial or divergent). Given the formula (35), it is merely a matter of technical analysis to obtain the limit. For completeness, this derivation is given in the appendix (section B), as it is a non-standard derivation. The final result is the well known formula for the price of the European call option under a log-normal assumption for the dynamics.

$$C(S(0), K, r, \sigma, T) = S \int_{-\infty}^{d_1} \frac{ds}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} - K e^{-rT} \int_{-\infty}^{d_2} \frac{ds}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} \quad (36)$$

where

$$d_1 = \frac{\log S/K + (r + \sigma^2/2)T}{\sqrt{\sigma^2 T}} \quad d_2 = \frac{\log S/K + (r - \sigma^2/2)T}{\sqrt{\sigma^2 T}} \quad (37)$$

An alternate approach is to obtain the limiting distribution of  $S(T)$  from (34). Taking this limit (see appendix section B) we get the well known log-normal distribution for the price of the stock.

$$p_{S(T)}(S(T)) = \frac{1}{S(T)\sqrt{2\pi\sigma^2 T}} \exp \left( -\frac{\log \frac{S(T)}{S(0)} - (r - \sigma^2/2)T}{2\sigma^2 T} \right) \quad (38)$$

Taking the expectation of  $(S(T) - K)\Theta(S(T) - K)$  with respect to this distribution is another approach to obtaining (36). We note that the  $\mu$  dependence has disappeared due to some “lucky” cancellations. We could have expected this, as the risk neutral dynamics (28) do not contain  $\mu$  dependence. Up to now, we have not invoked any stochastic calculus. The limit of the dynamics in (28) can be represented by the Ito stochastic differential equation

$$dS = rSdt + \sigma SdW \quad (39)$$

where (informally)  $dW$  is a random variable with variance  $dt$ . A standard application of Ito's lemma (see for example [13]) immediately yields (38), from which (36) follows by taking the risk neutral expectation.

There are many approaches to obtaining the price of the European call for the dynamics that we have chosen. For other dynamics, such closed form solutions are generally not possible, and one can resort to obtaining the prices using Monte Carlo techniques to take risk neutral expectations.

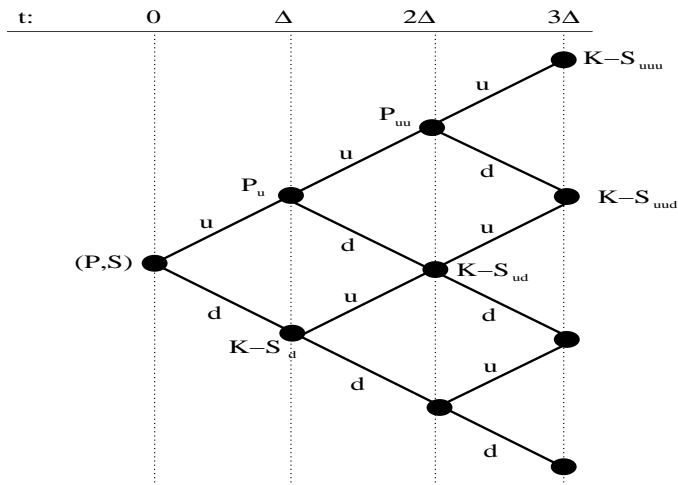
### 4.3 The American Put Option

An active area of research is the American put option, and no closed form expression for its price has been obtained for the stock bond dynamics that have already been described in the previous section. Some analytical series expressions exist (for example [15]), but the wealth of activity has been in efficient numerical techniques, namely PDE techniques (see for example [31], [30]) and Monte Carlo techniques (see for example [1], [3], [4], [20], [28], [29]). The *American put option* with expiry at time  $T$  and strike at  $K$  gives the holder the right (but not the obligation) to sell the stock at *any* time  $t \leq T$  at the strike price  $K$ . The key difference between the American and European option is the ability to sell at any time. The American call option is also defined in a similar way, except that instead of the right to sell, the holder has the the right to buy the stock. The American call and the European call have the same value (see for example [27]), hence we choose to study the put option here.

Before we study the American option, let us first introduce another instrument whose existence solely depends only on the existence of the put option itself, and hence we are justified in introducing it. Suppose the holder of the put decides at some time  $t$  to sell the put for the strike  $K$  when the stock is at  $S(t)$ . It seems sensible then that the (rational) holder would also have sold for all  $S < S(t)$  at time  $t$ . Thus we define an *exercise threshold function*,  $f_S(t)$ , as a function of  $t$  such that if at time  $t$ ,  $S(t) \leq f_S(t)$  then the holder exercises the option and if not, the holder will not exercise. The put option with a predetermined exercise threshold function is a *different* instrument from the original put option. Let's call this new instrument an *exercise predetermined American option*. The only difference between this instrument and the American option is that a specified exercise strategy must be followed. Some intuitive properties must hold:  $f_S(0) \leq S(0)$  and  $f_S(T) = K$ . We can ask what the price of this instrument should be. For illustration, suppose that  $f_S(t) = S(0)$ , for  $t \leq T$  and  $f_S(T) = K$ . Further suppose that we consider the 3 period economy of the previous section. We have already computed the risk neutral probability  $\tilde{P}_u$ . Let  $P(0, f_S(t))$  be the value of the exercise predetermined put at  $t = 0$ . For simplicity, assume that  $K > S_{uuu}$  and that  $\Delta$  is chosen small enough so that  $S_{udd} < S_{ud} < S(0)$  and  $S_d < S(0)$ . Let  $P_u(\Delta, f_S(t))$  be the value of holding this instrument if the state  $S_u$  is obtained and let  $P_d(\Delta, f_S(t))$  be the value of holding this instrument if the state  $S_d$  is obtained.  $P(0, f_S(t))$  is then given by the risk neutral expectation

$$P(0, f_S(t)) = e^{-r\Delta} \left( \tilde{P}_u P_u(\Delta, f_S(t)) + (1 - \tilde{P}_u) P_d(\Delta, f_S(t)) \right) \quad (40)$$

But, for our exercise threshold function, the put will be exercised if the state  $S_d$  is obtained so we know the value  $P_d(\Delta, f_S(t)) = K - S_d$ . One can iterate this argument to get  $P_u(\Delta, f_S(t))$ , and the continuation of this calculation is illustrated in the following lattice.



The final result for  $P_u(\Delta, f_S(t))$  is

$$e^{-r\Delta} \tilde{P}_d(K - S_d) + e^{-2r\Delta} \tilde{P}_d \tilde{P}_u(K - S_{ud}) + e^{-3\Delta} \tilde{P}_d \tilde{P}_u^2(K - S_{uud}) + e^{-3\Delta} \tilde{P}_u^3(K - S_{uuu}) \quad (41)$$

We can interpret this expression as follows. For all the possible  $2^3$  paths that the economy could take in this lattice model, compute what your payoff would have been, and discount it to time  $t = 0$  from the time at which you exercised the option. Then, weight this discounted payoff by the *risk neutral* probability of obtaining the path. We are thus computing the expected discounted cash-flow under risk neutral evolution of the economy. In principle we could have done this for any  $f_S(t)$ , providing that  $f_S(t)$  has been pre-specified.

We now come to the price  $P(0)$  of our American put option. For *any*  $f_S(t)$ ,  $P(0) \geq P(0, f_S(t))$ . To see this, suppose that this is not true, then an immediate type II arbitrage opportunity results from buying the American put, selling the exercise predetermined American put, and then exercising the American put according to  $f_S(t)$ . Since this inequality is true for any  $f_S(t)$  it must also hold when we maximize over all possible  $f_S(t)$ , assuming this maximum to exist. Further, since the holder of the American put must exercise according to some function, we have that

$$P(0) = \sup_{f_S(t)} P(0, f_S(t)) \quad (42)$$

where the sup is over all possible exercise functions. That this supremum can be taken only over exercise threshold functions of the form we have discussed will not be discussed further, though it is intuitively plausible. Thus, we see that pricing the American put has been reduced to a variational problem, and a large fraction of work in this area is devoted to developing efficient approximate solutions to this optimization problem using Monte Carlo techniques. However, the case is by no means closed.

Certainly, if one is to optimize over  $f_S(t)$ , one needs an efficient method for computing  $P(0, f_S(t))$ , especially since the optimization of  $f_S(t)$  is over  $N - 1$  parameters which is an exponentially growing problem. The approach we considered is too costly, for it involves evaluating  $2^N$  paths. An alternative is to Monte Carlo the paths according to the risk neutral measure and compute an estimate for the expectation of the discounted cash-flow by taking

an average<sup>6</sup>. Along these lines, one can proceed from time  $T$  backwards, optimizing at each time step using regression or neuro-dynamic programming type approaches (see [20], and [29]). Alternatively, one could maximize over parameterized forms for  $f_S(t)$ . Fortunately, the gradients needed for efficient optimization in the parameter space may be computed by Monte Carlo methods as well (see [9], [12], [10], [11]).

Finally, if for given  $f_S(t)$  one could analytically compute the exercise predetermined American put option price (either in the discrete case or in the limit of  $\Delta \rightarrow 0$ ), then, one would have a lower bound on  $P(0)$ , which could be tight if  $f_S(t)$  is realistic.

## 4.4 Interest Rate Derivatives

A spot roll-over or money-market account is an instrument whose value compounds continuously at the instantaneous interest rate. We allow the instantaneous interest rate to take on one of two possible values,  $r_1$  and  $r_2$ , with  $r_1 > r_2$ . Suppose that there also exists a 1 period zero coupon bond, that pays \$1 at the end of the period and that it is priced at  $B(1)$ . The economy is represented by

$$\mathbf{S}(0) = \begin{bmatrix} R \\ B(1) \end{bmatrix} \quad \mathbf{Z}(\Delta) = \begin{bmatrix} Re^{r_1\Delta} & Re^{r_2\Delta} \\ 1 & 1 \end{bmatrix} \quad (43)$$

Choosing the roll-over account as numeraire, one can compute the risk neutral probability of being in state  $r_1$  as

$$p = \tilde{P}_{r_1} = \frac{B(1) - e^{-r_2\Delta}}{e^{-r_1\Delta} - e^{-r_2\Delta}} \text{ provided that } e^{-r_2\Delta} \geq B(1) \geq e^{-r_1\Delta} \quad (44)$$

We now extend to  $N$  time periods, and consider the instrument that yields \$1 in every state at time  $t = N\Delta$ . The possible states of the roll-over account are  $Re^{\Delta(kr_1 + (N-k)r_2)}$  where  $k = 0, \dots, N$ . Thus

$$B(N) = E_{\tilde{\mathbf{P}}} \left[ e^{-\Delta(kr_1 + (N-k)r_2)} \right] = E_{\tilde{\mathbf{P}}} \left[ e^{-\Delta \sum r(i)} \right] \quad (45)$$

where  $r(i)$  represents the instantaneous interest rate at time period  $i$ . Converting the sum into an integral (in the limit  $\Delta \rightarrow 0$ ),

$$B(N) = E_{\tilde{\mathbf{P}}} \left[ e^{-\int_0^T dt r(t)} \right] \quad (46)$$

where the expectation is with respect to the risk neutral measure. This expression is generally true when one has chosen the roll-over account as numeraire. Continuing with the discrete time case, the expectation in (45) is over a binomial distribution (as  $\tilde{\mathbf{P}}$  is given by a binomial as in (34)) and is given by

$$B(N) = \sum_{k=0}^N \binom{N}{k} p^k (1-p)^{N-k} e^{-\Delta(kr_1 + (N-k)r_2)} \quad (47)$$

$$= B(1)^N \quad (48)$$

---

<sup>6</sup>This is commonly done by simulating discrete paths from the continuous risk neutral dynamics (39). However, more accurate is to use the risk neutral paths directly from the discrete dynamics (28), which comes at the expense of an additional parameter  $\mu$ .



A little insight would have led us directly to this result, since to guarantee \$1 at time period  $N$ , one buys the one period bond for  $B(1)$  at time  $N - 1$ , which requires  $B(1)$  one period bonds at time  $N - 2$ , costing  $B(1)^2$ , and so on. The key is that our derivation was purely mechanical. For more on interest rate modeling, the reader is referred to [26], [5].

## 5 Discussion

The purpose of this paper has been to provide a self contained exposition of the martingale approach to pricing that assumes a minimal level of finance or stochastic calculus. The reader interested in further reading is referred to the many excellent texts in the references. Several illustrative examples have been provided, some of them open problems even for the dynamics considered here, and, all of them are unresolved for increasingly complicated dynamics. The key is that while financial insight may be useful in addressing many of the examples, pricing can be achieved in a purely mechanical manner: obtain the risk neutral measure and then price using expectations with respect to this measure. Thus, the martingale approach is extremely well suited to the application of Monte Carlo techniques which provides a rich source of problems for the numerous Monte Carlo techniques available.

## A Proof of Positive Supporting Price Theorem

**Theorem A.1 (Positive supporting price)** *The following statements are equivalent.*

1. *There do not exist arbitrage opportunities of type I or type II.*
2. *There exists  $\psi > 0$  such that  $\mathbf{S}(0) = \mathbf{Z}(T)\psi$ .*

PROOF:

**(2) $\Rightarrow$ (1)** Suppose that  $\Theta$  is a portfolio with  $\Theta^T \mathbf{Z}(T) \geq 0$ , then,  $\Theta^T \mathbf{Z}(T)\psi > 0 \Rightarrow \Theta^T \mathbf{S}(0) > 0$  so there is no type I arbitrage. Suppose that  $\Theta$  is a portfolio with  $\Theta^T \mathbf{Z}(T) \geq 0$ , then,  $\Theta^T \mathbf{Z}(T)\psi \geq 0 \Rightarrow \Theta^T \mathbf{S}(0) \geq 0$  so there is no type II arbitrage.

**(1) $\Rightarrow$ (2)** Consider the following linear program

$$\mathbf{P1} : \quad \min_{\Theta} \mathcal{F}(\Theta) = \Theta^T \mathbf{S}(0) \quad s.t. \quad -\Theta^T \mathbf{Z}(T) \leq 0 \quad (49)$$

Immediately we observe that  $\Theta = 0$  is feasible and gives  $\mathcal{F}(\Theta) = 0$  therefore the value of  $\mathcal{F}$  at the solution to the linear program must be  $\leq 0$ . If it is  $< 0$  then we have a type II arbitrage opportunity. Thus, we conclude that the solution to this linear program exists, is finite and has value 0. By the duality theorem, the dual to this linear program is feasible (see for example [8]). The dual for this program is

$$\mathbf{P2} : \quad \max_{\psi} \mathcal{D}(\psi) = \mathbf{0}^T \psi \quad s.t. \quad \mathbf{Z}(T)\psi = \mathbf{S}(0) \quad (50)$$

$$\psi \geq 0$$

so we conclude that  $\exists \psi \geq 0$  such that  $\mathbf{S}(0) = \mathbf{Z}(T)\psi$ . If any constraint is not saturated at the minimum of (49), then there is a type I arbitrage opportunity, so we conclude that all

the constraints are saturated at the minimum, thus the dual must admit an interior feasible solution <sup>7</sup>. ■

## B The $\Delta \rightarrow 0$ Limit of the Binomial Model

In order to take the limit of (35) and to obtain the limiting distribution of (34) we need to analyse the binomial type expression

$$\mathcal{B}(k, \alpha_+, \alpha_-) = \binom{N}{k} \alpha_+^k \alpha_-^{N-k} \quad (51)$$

The well known Normal approximation to the binomial (see for example [7]) gives

$$\mathcal{B}(k, \alpha_+, \alpha_-) \rightarrow \frac{(\alpha_+ + \alpha_-)^N}{\sqrt{\frac{2\pi N\alpha_+\alpha_-}{(\alpha_+ + \alpha_-)^2}}} \exp\left(-\frac{\left(k - \frac{N\alpha_+}{\alpha_+ + \alpha_-}\right)^2}{\frac{2N\alpha_+\alpha_-}{(\alpha_+ + \alpha_-)^2}}\right) \quad (52)$$

The  $\Theta(\cdot)$  in (35) restricts  $k$  in the summation to

$$k \geq \frac{\log \frac{K}{S(0)} - N \log \lambda_-}{\log \frac{\lambda_+}{\lambda_-}} \rightarrow \frac{\log \frac{K}{S} + N\sigma\sqrt{\Delta} - N\Delta(\mu - \frac{1}{2}\sigma^2)}{2\sigma\sqrt{\Delta}} \quad (53)$$

Converting the summation to an integral in (35) and extending the upper limit to  $\infty$ , one arrives at

$$e^{rT}C(0) = S(0) \int_L^\infty dk \mathcal{B}(k, \lambda_+ \tilde{P}_u, \lambda_- (1 - \tilde{P}_u)) - K \int_L^\infty dk \mathcal{B}(k, \tilde{P}_u, 1 - \tilde{P}_u) \quad (54)$$

where in both cases, the lower limit of integration,  $L$ , is given by (53).  $\lambda_\pm$  is given in (23) and  $\tilde{P}_u$  is given in (25). A change of variables can be applied to convert the integrals into the form we are after:

$$\int_L^\infty dk \mathcal{B}(k, \alpha_+, \alpha_-) = \int_{\left(L - \frac{N\alpha_+}{\alpha_+ + \alpha_-}\right) \frac{\alpha_+ + \alpha_-}{\sqrt{N\alpha_+\alpha_-}}}^\infty \frac{du}{\sqrt{2\pi}} (\alpha_+ + \alpha_-)^N e^{-\frac{1}{2}u^2} \quad (55)$$

Note that  $(\lambda_+ \tilde{P}_u + \lambda_- (1 - \tilde{P}_u))^N \rightarrow (1 + r\Delta)^N \rightarrow e^{rT}$ . Taking the limit as  $\Delta \rightarrow 0$  for the lower limits in our two integrals yields  $-d_1$  and  $-d_2$  respectively. Since switching the limits of the integration and changing their signs leaves the integral unchanged, we are led to (36).

A similar approach yields the distribution (38).

$$P[S \leq S(T)] = P\left[k \leq \frac{\log \frac{S(T)}{S(0)} - N \log \lambda_-}{\log \frac{\lambda_+}{\lambda_-}}\right] \quad (56)$$

$$= \int^U dk \mathcal{B}(k, \tilde{P}_u, 1 - \tilde{P}_u) \quad (57)$$

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<sup>7</sup>alternatively, set up the Lagrangian  $\mathcal{L} = \Theta^T \mathbf{S}(0) - \Theta^T \mathbf{Z}(T) \lambda$  where  $\lambda$  is the set of Lagrange multipliers. By the *Kuhn – Tucker* theorem, we need to minimize over  $\Theta$  and maximize over  $\lambda \geq 0$ . Since we know that a solution exists, there must exist  $\lambda^* \geq 0$  for which  $\partial \mathcal{L}(\Theta, \lambda^*) / \partial \Theta = \mathbf{S}(0) - \mathbf{Z}(T) \lambda^* = 0$ . Since all the constraints are saturated at the minimum, an interior solution for  $\lambda$  exists.

where  $U$  is given by (56). Taking the derivative with respect to  $S(T)$  gives

$$p_{S(T)}(S(T)) = \frac{1}{S(T) \log \frac{\lambda_+}{\lambda_-}} \mathcal{B}(U, \tilde{P}_u, 1 - \tilde{P}_u) \quad (58)$$

which upon taking the limit  $\Delta \rightarrow 0$  gives (38).

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