

Efficient Computation of Optimal Trading Strategies

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Abstract

Given the return series for a set of instruments, a *trading strategy* is a switching function that transfers wealth from one instrument to another at specified times. We present efficient algorithms for constructing (ex-post) trading strategies that are optimal with respect to the total return, the Sterling ratio and the Sharpe ratio. Such optimal strategies are useful as benchmarks, and for identifying the optimal trades that can be used to to teach (ex-ante) predictors.

1 Introduction

A trader has in mind the task of developing a trading system that optimizes some profit criterion, the simplest being the total return. A more conservative approach is to optimize a risk adjusted return. Widely followed measures of risk adjusted returns are the Sterling Ratio and Sharpe Ratio. In an environment where markets exhibit frequent crashes and portfolios encounter sustained periods of losses, the Sterling ratio and the Sharpe ratio have emerged as the leading performance measures used in the industry. Given a set of instruments, a trading strategy is a switching function that transfers the wealth from one instrument to another. In this paper, we consider the problem of finding optimal trading strategies, i.e., trading strategies that maximize a given optimality criterion. In particular, we consider optimal strategies with respect to the total cumulative return, as well as with respect to various risk adjusted measures of return (the Sterling ratio and variants of the Sharpe ratio). A brute force approach to obtaining such optimal trading strategies would search through the space of all possible trading strategies, keeping only the one satisfying the optimality criterion. Since the number of possible trading strategies grows exponentially with time, the brute

force approach leads to an exponential time algorithm¹, which for all practical purposes is infeasible – even given the pace at which computing power grows. The contribution in this work is to give *efficient* (polynomial time) algorithms to compute the optimal trading strategy for various profit objectives, and under constraints on the number of trades that can be made.

Our motivations for constructing such optimal strategies are: (i) Knowing what the optimal trades are, one can try to *learn* to predict good trading opportunities by using market and/or technical indicators as features on which to base the prediction. A host of such activity goes under the name of *financial engineering*. (ii) The optimal performance modulo certain trading constraints can be used as a benchmark for real trading systems. For example, how good is a trading system that makes ten trades with a Sterling ratio of 4 over a given time period? One natural comparison is to benchmark this trading strategy against a Sterling-optimal trading strategy that makes at most ten trades over the same time period. (iii) Optimal trading strategies (with or without constraints) can be used to quantitatively rank various markets (and time scales) with respect to their profitability according to a given criterion. So for example, one could determine the optimal time scale on which to trade a particular market, or given a set of markets, which is the most (risk adjusted) profit-friendly.

In order to make the preceding discussion more precise and to more accurately state our results, let's introduce a few definitions. Assume that we have two instruments, for concreteness, a stock S and a bond B with price histories $\{S_0, \dots, S_n\}$ and $\{B_0, \dots, B_n\}$ over n consecutive time periods, $t_i, i \in \{1, \dots, n\}$. Thus, for example, over time period t_i , the price of stock moved from S_{i-1} to S_i . We denote the return sequence for the two instruments by $\{s_1, \dots, s_n\}$ and $\{b_1, \dots, b_n\}$ respectively: $s_i = \log \frac{S_i}{S_{i-1}}$, and correspondingly, $b_i = \log \frac{B_i}{B_{i-1}}$. We assume that one of the instruments is the benchmark instrument, and that all the equity is held in the benchmark instrument at the beginning and end of trading. The bond is usually considered the benchmark instrument, and for illustration, we will follow this convention. The trivial trading strategy is to simply hold onto bond for the entire duration of the trading period. It is useful to define the excess return sequence for the stock, $\hat{s}_i = s_i - b_i$. When the benchmark instrument is the bond, the excess return as we defined it is the conventionally used one. However, one may want to measure performances of a trading strategy with respect to the S&P 500 as benchmark instrument, in which case the excess return would be determined relative to the S&P 500 return sequence. The excess return sequence for the bond is just the sequence of zeros, $\hat{b}_i = 0$. Conventionally, the performance of a strategy is measured relative to some trivial strategy, so the excess return sequence will be the basis of most of our performance measures.

¹The asymptotic running time of an algorithm is measured in terms of the input size n . If the input is a time sequence of n price data points, then polynomial time algorithms have run time that is bounded by some polynomial in n . Exponential time algorithms have running time greater than some exponentially growing function in n [7].

Definition 1.1 (Trading Strategy) A trading strategy \mathcal{T} is a boolean n -dimensional vector indicating where the money is at the end of time period t_i :

$$\mathcal{T}[i] = \begin{cases} 1 & \text{if money is in stock at the end of } t_i, \\ 0 & \text{if money is in bond at the end of } t_i. \end{cases}$$

We assume that $\mathcal{T}[0] = \mathcal{T}[n] = 0$, i.e., all the money begins and ends in bond. A trade is entered at time t_i if $\mathcal{T}[i] = 0$ and $\mathcal{T}[i+1] = 1$. A trade is exited at time t_i if $\mathcal{T}[i] = 1$ and $\mathcal{T}[i+1] = 0$. The number of trades made by a trading strategy is equal to the number of trades that are entered.

Note that we make the following assumptions regarding the trading:

- A1** [All or Nothing]: The position at all times is either entirely bond or entirely stock.
- A2** [No Market Impact]: Trades can be placed without affecting the quoted price.
- A3** [Fractional Market]: Arbitrary amounts of stock or bond can be bought or sold at any time.
- A4** [Long Strategies]: We assume that we can only hold long positions in stock or bond.

These assumptions are rather mild and quite accurate in most liquid markets, for example foreign exchange. **A1** implies (for example) that one can not leg into a trade. For some optimality criteria, legging into a trade may be beneficial, however, in most circumstances, an all-or-nothing optimal strategy can be chosen. **A3** is a consequence of **A1**, since if all the money should be transferred to a stock position, this may necessitate the purchase of a fractional number of shares. Note that if $\mathcal{T}[i-1] \neq \mathcal{T}[i]$, then at the beginning of time period t_i , the position was transferred from one instrument to another. Such a transfer will incur an instantaneous per unit transaction cost equal to the bid-ask spread of the instrument being transferred into. We assume that the bid-ask spread is some fraction (f_B for bond and f_S for stock) of the bid price.

We denote the equity curve for a trading strategy \mathcal{T} by the vector $\mathcal{E}_{\mathcal{T}}$, i.e., $\mathcal{E}_{\mathcal{T}}[i]$ is the equity at the end of time period t_i , with $\mathcal{E}_{\mathcal{T}}[0] = 1$. Corresponding to the equity curve is the excess return sequence $r_{\mathcal{T}}$ for the trading strategy \mathcal{T} , i.e., for $i \geq 1$

$$r_{\mathcal{T}}[i] = \log \frac{\mathcal{E}_{\mathcal{T}}[i]}{\mathcal{E}_{\mathcal{T}}[i-1]} - b_i. \quad (1)$$

If we ignore the bid-ask spread, then the excess return in time period t_i is given by

$$r_{\mathcal{T}}[i] = \hat{s}_i \mathcal{T}[i] = (s_i - b_i) \mathcal{T}[i]. \quad (2)$$

The bid-ask spread affects the return, reducing it by an amount depending on $\mathcal{T}[i-1]$. Denoting this transactions cost attributable to $\mathcal{T}[i]$ by $\Delta[i]$, we have that

$$\Delta[i] = -\mathcal{T}[i-1](1 - \mathcal{T}[i])\hat{f}_B - (1 - \mathcal{T}[i-1])\mathcal{T}[i]\hat{f}_S, \quad (3)$$

where $\hat{f}_S = \log(1 + f_S)$ and $\hat{f}_B = \log(1 + f_B)$. Thus, the bid-ask spread can be viewed as introducing an instantaneous return of $-\hat{f}_B$ or $-\hat{f}_S$ whenever the position is switched. To exactly which time period this transactions cost is applied may depend on the nature of the market, i.e., it may be applied to $r_{\mathcal{T}}[i]$, $r_{\mathcal{T}}[i - 1]$ or $r_{\mathcal{T}}[i + 1]$. The nature of the results will not change significantly for either of these options, so in our algorithms, we will generally make the choice that offers the greatest technical simplicity. For a trading strategy \mathcal{T} , we define the total return $\mu(\mathcal{T})$, the sum of the squared returns $s^2(\mathcal{T})$, the sum of squared deviations of the returns $\sigma^2(\mathcal{T})$ and the maximum drawdown $MDD(\mathcal{T})$ as follows,

$$\mu(\mathcal{T}) = \sum_{i=1}^n r_{\mathcal{T}}[i], \quad (4)$$

$$s^2(\mathcal{T}) = \sum_{i=1}^n r_{\mathcal{T}}[i]^2, \quad (5)$$

$$\sigma^2(\mathcal{T}) = \sum_{i=1}^n \left(r_{\mathcal{T}}[i] - \frac{1}{n} \mu(\mathcal{T}) \right)^2 = s^2(\mathcal{T}) - \frac{1}{n} \mu^2(\mathcal{T}), \quad (6)$$

$$MDD(\mathcal{T}) = \max_{1 \leq k \leq l \leq n} - \sum_{i=k}^l r_{\mathcal{T}}[i]. \quad (7)$$

When it is clear from the context what trading strategy we are talking about, we will generally suppress the explicit dependence on \mathcal{T} . The performance measures that we consider in this paper are derived from these statistics. In particular, we are interested in the total return μ , the Sterling ratio Strl , and variants of the Sharpe ratio, Shrp_1 and Shrp_2 :

$$\text{Strl}(\mathcal{T}) = \frac{\mu(\mathcal{T})}{MDD(\mathcal{T})}, \quad \text{Shrp}_1(\mathcal{T}) = \frac{\mu(\mathcal{T})}{\sigma(\mathcal{T})}, \quad \text{Shrp}_2(\mathcal{T}) = \frac{\mu(\mathcal{T})}{\sigma^2(\mathcal{T})}. \quad (8)$$

Shrp_1 is the conventionally used Sharpe ratio. Shrp_2 is a more risk averse performance measure, as it is more sensitive to the variance in the returns. Often, Strl as we have defined it is referred to as the Calmar ratio in the literature [11], and the Sterling ratio adds a constant (for example 10%) to the MDD in the denominator [1]. Such a constant can easily be accommodated by our algorithms, and so we will maintain this simpler definition for the Sterling ratio.

The contribution of this paper is efficient algorithms for computing optimal trading strategies. We will use standard $O()$ notation in stating our results: let n be the length of the returns sequences; we say that the run time of an algorithm is $O(f(n))$ if, for some constant C , the runtime is $\leq Cf(n)$ for any possible return sequences. If $f(n)$ is linear (quadratic), we say that the runtime is linear (quadratic). We will establish the following results.

Theorem 1.2 (Return Optimal Trading Strategies) *A total return optimal trading strategy*

can be computed in linear time. Specifically,

- i. **Unconstrained Trading.** A trading strategy \mathcal{T}_μ can be computed in $O(n)$ such that for any other strategy \mathcal{T} , $\mu(\mathcal{T}_\mu) \geq \mu(\mathcal{T})$.
- ii. **Constrained Trading.** A trading strategy \mathcal{T}_μ^K making at most K trades can be computed in $O(K \cdot n)$ such that for any other strategy \mathcal{T}^K making at most K trades, $\mu(\mathcal{T}_\mu^K) \geq \mu(\mathcal{T}^K)$.

Proof: See section 2. ■

Theorem 1.3 (Sterling Optimal Trading Strategies) A Sterling optimal trading strategy can be computed in near linear time. Specifically,

- i. **Unconstrained Trading.** A trading strategy $\mathcal{T}_{\text{Strl}}$ can be computed in $O(n \log n)$ such that for any other strategy \mathcal{T} , $\text{Strl}(\mathcal{T}_{\text{Strl}}) \geq \text{Strl}(\mathcal{T})$.
- ii. **Constrained Trading.** A trading strategy $\mathcal{T}_{\text{Strl}}^K$ making at most K trades can be computed in $O(n \log n)$ such that for any other strategy \mathcal{T}^K making at most K trades, $\text{Strl}(\mathcal{T}_{\text{Strl}}^K) \geq \text{Strl}(\mathcal{T}^K)$.

Proof: See section 3. ■

Theorem 1.4 (Sharpe Optimal Trading Strategies) A Sharpe optimal trading strategy can be computed in near quadratic time. Specifically, trading strategies $\mathcal{T}_{\text{Shrp}_1}$ and $\mathcal{T}_{\text{Shrp}_2}$ can be found in $O(n^2 \log n)$ such that for any other strategy \mathcal{T} , $\text{Strl}_1(\mathcal{T}_{\text{Shrp}_1}) \geq \text{Strl}_1(\mathcal{T})$ and $\text{Strl}_2(\mathcal{T}_{\text{Shrp}_2}) \geq \text{Strl}_2(\mathcal{T})$

Proof: See section 4. ■

In all cases, our proofs are constructive, and so immediately give the algorithms for performing the desired computations. Next, we discuss the existing related work, followed by a detailed discussion of the algorithms, along with all necessary proofs.

Related Work

The body of literature on optimal trading is so enormous that we only mention here some representative papers. All the research on optimal trading falls into two broad categories. The first group is on the more theoretical side where researchers assume that stock prices satisfy some particular model, for example the prices are driven by a stochastic process of known form; the goal is to derive closed-form solutions for the optimal trading strategy, or a set of equations that the optimal strategy must follow. We highlight some of this research below.

Lobo et al. in [16] consider the problem of single-period portfolio optimization. They consider the maximization of the expected return subject to different types of constraints on the portfolio

(margin, diversification, budget constraints and limits on variance or shortfall risk). Under the assumption that the return on asset i at the end of period is the random variable a_i and both first and second moments of the joint distribution of the random variables are known, this optimization problem can be represented as a convex optimization problem and thus can be efficiently solved by a numerical methods (eq. by Karmarkar's interior-point method [14]).

Thompson in [24] considered the problem of maximizing the (expected) total cumulative return of a trading strategy under the assumption that the asset price satisfies a stochastic differential equation of the form $dS_t = dB_t + h(X_t)dt$, where B_t is a Brownian motion, h is a known function and X_t is a Markov Chain independent of the Brownian motion. In this work, he assumes fixed transaction costs and imposes assumptions $A1$, $A2$, $A4$ on the trading. He also imposes a stricter version of our assumption $A3$: at any time, trader can have only 0 or 1 unit of stock. He proves that the optimal trading strategy is the solution of a free-boundary problem, gives explicit solutions for several functions h and provides bounds on the transaction cost above which it is optimal never to buy the asset at all.

Pliska et al. in [4] considered the problem of an optimal investment for a continuous-time market consisting of the usual bank account, a rolling horizon bond and a discount bond whose maturity coincides with the planning horizon. They assume interest rates to be stochastic (driven by a stochastic differential equation) and derive an equation satisfied by the trading strategy that maximizes the HARA utility wealth function.

Bielecki in [3] considered the problem of maximizing the risk sensitive expected exponential growth rate of the investor's portfolio in a economy model consisting of a bank account and a risky security (stock) with a stochastic interest rate. The optimal trading strategy is characterized in terms of a nonlinear quasi-variational inequality and he developed a numerical approach to solving this equation.

Berkelaar and Kouwenberg in [2] considered asset allocation in a mean versus downside-risk framework. Downside-risk measures penalize only negative returns relative to a given benchmark. Investors have a trade-off between mean and downside-risk. Closed-form solutions for the optimal asset allocation are derived for the case where asset prices follow geometric Brownian motions with constant interest rate.

The main drawbacks of such theoretical approaches is that their prescriptions can only be useful to the extent that the assumed models are correct. Our work does not make any assumptions about the price dynamics to construct ex-post optimal trading strategies. The second group of research which is more on the practical side is focused on exploring learning methods for the prediction of future stock prices moves. Intelligent agents are designed by training on past data and their performance is compared with some benchmark strategies. Our results furnish (i) optimal strategies on which to train intelligent agents and (ii) benchmarks with which to compare their performance.

Liu in [15] consider the optimal investment policy of a constant absolute risk aversion (CARA) investor who faces fixed and proportional transaction costs when trading multiple risky assets. He show that when asset returns are uncorrelated, the optimal investment policy is to keep the dollar amount invested in each risky asset between two constant levels and upon reaching either of these thresholds, to trade to the corresponding optimal targets.

Zakamouline in [27] studies the optimal portfolio selection problem for a constant relative risk averse investor who faces fixed and proportional transaction costs and maximizes expected utility of the investor's end-of-period wealth. The author applies the method of the Markov chain approximation to numerically solve for the optimal trading strategy. The numerical solution indicates that the portfolio space is divided into three disjoint regions (Buy, Sell and No-Transaction), and four boundaries describe the optimal strategy. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the lower (Buy) target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the upper (Sell) target boundary. All these boundaries are functions of the investor's horizon and the composition of the investor's wealth.

Choi and Liu in [20] considered trading tasks faced by an autonomous trading agent. An autonomous trading agent works as follows. First, it observes the state of the environment. According to the environment state, the agent responds with an action, which in turn influences the current environment state. In the next time step, the agent receives a feedback (reward or penalty) from the environment and then perceives the next environment state. The optimal trading strategy for the agent was constructed in terms of the agent's expected utility (expected accumulated reward).

Cuoco et al. in [9] considered Value at Risk as a tool to measure and control the risk of the trading portfolio. The problem of a dynamically consistent optimal portfolio choice subject to the Value at Risk limits was formulated and they proved that the risk exposure of a trader subject to a Value at Risk limit is always lower than that of an unconstrained trader and that the probability of extreme losses is also decreased. They also prove that another risk measure - Tail Conditional Expectation - is equivalent to the Value at Risk. In particular, they showed that in a dynamic setting it is always possible to transform any given Tail Conditional Expectation limit into an equivalent Value at Risk limit and conversely.

Dammon and Spatt in [19] explored the optimal trading and pricing of taxable securities with asymmetric capital gains taxes and transaction costs. Under current U.S. tax law, gains on capital assets are not taxed until the investor sells the asset and the tax rate that applies to capital gains and losses may be a function of the investor's holding period. These features give investors the incentive to time their asset sales so as to minimize the tax burden of owning taxable securities. In this work the optimal trading strategy is derived in the presence of capital gains taxes and transaction costs and the implications of the optimal trading strategy for asset pricing is explored.

Mihatsch and Neuneier in [17] considered problem of optimization of a risk-sensitive expected return of a Markov Decision Problem. Based on an extended set of optimality equations, risk-sensitive versions of various well-known reinforcement learning algorithms were formulated and they showed that these algorithms converge with probability one under reasonable conditions.

The Sharpe ratio was introduced in [21], [22] and since then became a popular and widespread risk-sensitive measure of a portfolio performance. Faugere et al. in [6] used Sharpe ratio as one of performance measures to compare effectiveness of different decision-making criteria. Pedersen et al. in [18] performed empirical comparison of many performance measurement methodologies for the global financial services sector and documented strong evidence in support of using Sharp-Ratio based measures. It was also pointed out that correlation of the Sterling ratio with other measures is usually small and dips below 5% in some instances.

A number of authors associated with BARRA (a major supplier of analytic tools and databases) have used the terms *information ratio* [10] or *Appraisal ratio* instead [25]. Goodwin in [10] considers the relationship between the Sharpe ratio and other performance measures, compares four methods of annualizing an information ratio, and presents the empirical evidence on the distribution of information ratios by style, which provides a context in which to examine manager performance.

Surprisingly, there is little previous research on portfolio optimization with respect to the Sharpe ratio or Sterling ratio, partially because of the intrinsic difficulty of these non-linear optimization criteria.

Moody and Saffell in [13] presented methods for optimizing portfolios, asset allocations and trading systems based on a direct reinforcement approach. In this approach, investment decision making is viewed as a stochastic control problem and the need to build forecasting models is eliminated. An adaptive algorithm called recurrent reinforcement learning for discovering investment policies was proposed and they demonstrated how it can be used to optimize risk-adjusted investment returns like the Sterling Ratio or Sharpe Ratio, while accounting for the effects of transaction costs.

Liu et al. in [26] proposed a learning-based trading strategy for portfolio management, which aims at maximizing the Sharpe Ratio by actively reallocating wealth among assets. The trading decision is formulated as a non-linear function of the latest realized asset returns, and the function can be approximated by a neural network. In order to train the neural network, one requires a Sharpe-Optimal trading strategy to provide the supervised learning method with target values. In this work they used heuristic methods to obtain a locally Sharp-optimal trading strategy. The transaction cost was not taken into consideration. Our methods can be considerably useful in the determination of target trading strategies for such approaches.

Tasche and Tibiletti in [23] explored ways to speed up the online computation of the Sharpe ratio of a current portfolio and how the Sharpe ratio will change if a candidate new asset will be

incorporated into the portfolio. Approximation formulae were derived that are based on certain derivatives of the Value-at-Risk.

Hellstrom in [12] formulates an alternative formulation of the stock prediction problem based on a statistical investigation of the stock's ranks. In this work a single perceptron neural network is trained to find the parameter vector that maximizes the Sharpe ratio. Due to the lack of the Sharpe-optimal trading decisions that can be supplied as target values, a special technique for optimization without derivatives is utilized [8].

2 Return-Optimal Trading Strategies

We use the notation $[t_i, t_j]$ to denote the set of time periods $\{t_i, t_{i+1}, \dots, t_j\}$. In order to compute the return optimal strategies, we will use a dynamic programming approach to solve a more general problem. Specifically, we will construct the return optimal strategies for every prefix of the returns sequence. First we consider the case when there is no restriction on the number of trades, and then the case when the number of trades is constrained to be at most K . Although we maintain assumptions **A1-A4** for simplicity, **A1**, **A3** and **A4** can be relaxed without much additional effort.

2.1 Unconstrained Return-Optimal Trading Strategies

First we give the main definitions that we will need in the dynamic programming algorithm to compute the optimal strategy. Consider a return-optimal strategy for the first m time periods, $[t_1, t_m]$. Define $\mathcal{S}[m, 0]$ ($\mathcal{S}[m, 1]$) to be a return-optimal strategy over the first m periods ending in bond (stock) at time t_m . For $\ell \in \{0, 1\}$, let $\mu[m, \ell]$ denote the return of $\mathcal{S}[m, \ell]$ over $[t_1, t_m]$, i.e., $\mu[m, \ell] = \mu(\mathcal{S}[m, \ell])$. Let $\text{PREV}[m, \ell]$ denote the penultimate position of the optimal strategy $\mathcal{S}[m, \ell]$ just before the final time period t_m .

The optimal strategy $\mathcal{S}[m, \ell]$ must pass through either bond or stock at time period $m - 1$. Thus, $\mathcal{S}[m, \ell]$ must be the extension of one of the optimal strategies $\{\mathcal{S}[m - 1, 0], \mathcal{S}[m - 1, 1]\}$ by adding the position ℓ at time period t_m . More specifically, $\mathcal{S}[m, \ell]$ will be the extension that yields the greatest total return. Using (2) and (3), we have that

$$\begin{aligned}\mu(\{\mathcal{S}[m - 1, 0], \ell\}) &= \mu[m - 1, 0] + \hat{s}_m \ell - \hat{f}_S \ell, \\ \mu(\{\mathcal{S}[m - 1, 1], \ell\}) &= \mu[m - 1, 1] + \hat{s}_m \ell - \hat{f}_B(1 - \ell).\end{aligned}$$

Since $\mu[m, \ell]$ is the maximum of these two values, we have the following recursion,

$$\mu[m, \ell] = \max \left\{ \mu[m - 1, 0] + \hat{s}_m \ell - \hat{f}_S \ell, \mu[m - 1, 1] + \hat{s}_m \ell - \hat{f}_B(1 - \ell) \right\}.$$

The position of the optimal strategy $\mathcal{S}[m, \ell]$ just before time period m is given by the ending position of the strategy that was extended. Thus,

$$\text{PREV}[m, \ell] = \begin{cases} 0 & \text{if } \mu[m-1, 0] + \hat{s}_m \ell - \hat{f}_S \ell \geq \mu[m-1, 1] + \hat{s}_m \ell - \hat{f}_B(1 - \ell), \\ 1 & \text{otherwise.} \end{cases}$$

If we already know $\mu[m-1, 0]$ and $\mu[m-1, 1]$, then we can compute $\mu[m, \ell]$ and $\text{PREV}[m, \ell]$ for $\ell \in \{0, 1\}$ in constant time. Further, we have that $\mu[1, 0] = 0$ and $\mu[1, 1] = \hat{s}_1 - \hat{f}_S$, and so, by a straight forward induction, we can prove the following lemma.

Lemma 2.1 *PREV[m, ℓ] for all $\ell \in \{0, 1\}$ and $m \leq n$ can be computed in $O(n)$.*

The optimal strategy \mathcal{T}_μ is exactly $\mathcal{S}[n, 0]$. $\text{PREV}[n, 0]$ gives the position at t_{n-1} , and the optimal way to reach $\text{PREV}[n, 0]$ at t_{n-1} is given by optimal strategy $\mathcal{S}[n-1, \text{PREV}[n, 0]]$. Continuing backward in this fashion, it is easy to verify that we can reconstruct the full strategy \mathcal{T}_μ using the following backward recursion:

$$\begin{aligned} \mathcal{T}_\mu[n] &= 0, \\ \mathcal{T}_\mu[m] &= \text{PREV}[m+1, \mathcal{T}_\mu[m+1]], \text{ for } 1 \leq m < n. \end{aligned}$$

Thus, a single backward scan is all that is required to compute $\mathcal{T}_\mu[i]$ for all $i \in \{1, \dots, n\}$, which is linear time, and so we have proved the first part of Theorem 1.2. Further, it is clear that the algorithm requires memory that is linear in n to store $\text{PREV}[m, \ell]$. While we have assumed that the algorithm works with excess returns, the optimal strategy does not depend on this assumption, thus the algorithm works correctly even with the actual return sequences. The generalization of this algorithm to $N > 2$ instruments is straightforward by suitably generalizing a trading strategy. $\mathcal{S}[m, \ell]$ retains its definition, except now $\ell \in \{0, \dots, N-1\}$. To compute $\mu[m, \ell]$ will need to take a maximum over N terms depending on $\mu[m-1, \ell']$, and so the algorithm will have runtime $O(Nn)$.

One concern with the unconstrained optimal strategy is that it may make too many trades. It is thus useful to compute the optimal strategy that makes at most a given number of trades. We discuss this next.

2.2 Constrained Return-Optimal Strategies

We suppose that the number of trades is constrained to be at most K . It is more convenient to consider the number of jumps k , which we define as the sum of the number of trades entered and the number exited. For a valid trading strategy, the number of trades entered equals the number of trades exited, so $k = 2K$. Analogous to $\mathcal{S}[m, \ell]$ in the previous section, we define $\mathcal{S}[m, k, \ell]$ to be

the optimal trading strategy to time period t_m that makes at most k jumps ending in instrument ℓ . Let $\mu[m, k, \ell]$ be the return of strategy $\mathcal{S}[m, k, \ell]$, and let $\text{PREV}[m, k, \ell]$ store the pair (k', ℓ') , where ℓ' is the penultimate position of $\mathcal{S}[m, k, \ell]$ at t_{m-1} that leads to the end position ℓ , and k' is the number of jumps made by the optimal strategy to time period t_{m-1} that was extended to $\mathcal{S}[m, k, \ell]$.

The algorithm once again follows from the observation that the the optimal strategy $\mathcal{S}[m, k, \ell]$ must pass through either bond or stock at t_{m-1} . A complication is that if the penultimate position is bond and $\ell = 0$, then at most k jumps can be used to get to thhe penultimate position, however, if $\ell = 1$, then only at most $k - 1$ jumps may be used. Similarly if the penultimate position is stock. We thus get the following recursion,

$$\begin{aligned}\mu[m, k, 0] &= \max \left\{ \mu[m-1, k, 0], \mu[m-1, k-1, 1] - \hat{f}_B \right\}, \\ \mu[m, k, 1] &= \max \left\{ \mu[m-1, k-1, 0] + \hat{s}_m - \hat{f}_S, \mu[m-1, k, 1] + \hat{s}_m \right\}.\end{aligned}$$

This recursion is initialized with $\mu[m, 0, 0] = 0$ and $\mu[m, 0, 1] = \text{NULL}$ for $1 \leq m \leq n$. Once $\mu[m, k, \ell]$ is computed for all m, ℓ , then the above recursion allows us to compute $\mu[m, k+1, \ell]$ is computed for all m, ℓ . Thus, the computation of $\mu[m, k, \ell]$ for $1 \leq m \leq n$, $0 \leq k \leq 2K$ and $\ell \in \{0, 1\}$ can be accomplished in $O(nK)$. Once again, the strategy that was extended gives $\text{PREV}[m, k, \ell]$,

$$\begin{aligned}\text{PREV}[m, k, 0] &= \begin{cases} (k, 0) & \text{if } \mu[m-1, k, 0] > \mu[m-1, k-1, 1] - \hat{f}_B, \\ (k-1, 1) & \text{otherwise.} \end{cases} \\ \text{PREV}[m, k, 1] &= \begin{cases} (k-1, 0) & \text{if } \mu[m-1, k-1, 0] + \hat{s}_m - \hat{f}_S > \mu[m-1, k, 1] + \hat{s}_m, \\ (k, 1) & \text{otherwise.} \end{cases}\end{aligned}$$

Since computing $\mu[m, k, \ell]$ immediately gives $\text{PREV}[m, k, \ell]$, we have the following lemma,

Lemma 2.2 $\text{PREV}[m, k, \ell]$ for all $\ell \in \{0, 1\}$, $m \leq n$ and $k \leq 2K$ can be computed in $O(nK)$.

T_μ^K is given by $S[n, 2K, 0]$, and the full strategy can be reconstructed in a single backward scan using the following backward recursion (we introduce an auxilliary vector κ),

$$\begin{aligned}\mathcal{T}_\mu^K[n] &= 0, \\ (\kappa[n-1], \mathcal{T}_\mu^K[n-1]) &= \text{PREV}[n, 2K, \mathcal{T}_\mu^K[n]], \\ (\kappa[m], \mathcal{T}_\mu^K[m]) &= \text{PREV}[m+1, \kappa[m+1], \mathcal{T}_\mu^K[m+1]], \text{ for } 1 \leq m < n-1.\end{aligned}$$

Since the algorithm needs to store $\text{PREV}[m, k, \ell]$ for all m, k , the memory requirement is $O(nK)$.

Once again, it is not hard to generalize this algorithm to work with N instruments, and the resulting run time will be $O(nNK)$.

3 Sterling-Optimal Trading Strategies

It will first be useful to discuss some of the MDD properties of the return-optimal strategy \mathcal{T}_μ , as these properties will have implications on our algorithm to determine Sterling-optimal strategies. For a strategy \mathcal{T} , it is useful to define the cumulative return series, $C_{\mathcal{T}}[i]$ as the sum of the returns, $C_{\mathcal{T}}[i] = \sum_{j=1}^i r_{\mathcal{T}}[j]$. Note that $\mu(\mathcal{T}_\mu) = C_{\mathcal{T}_\mu}[n] \geq C_{\mathcal{T}}[n] = \mu(\mathcal{T})$ for any strategy \mathcal{T} . The equity curve is given by $\mathcal{E}_{\mathcal{T}}[i] = \exp\left(C_{\mathcal{T}}[i] + \sum_{j=1}^i b_j\right)$.

First, we will upper bound $MDD(\mathcal{T}_\mu)$, because this in turn serves as an upper bound for the MDD of the Sterling-optimal strategy,

Lemma 3.1 $MDD(\mathcal{T}_{\text{Strl}}) \leq MDD(\mathcal{T}_\mu)$.

Proof: By definition, $\frac{\mu(\mathcal{T}_{\text{Strl}})}{MDD(\mathcal{T}_{\text{Strl}})} \geq \frac{\mu(\mathcal{T})}{MDD(\mathcal{T})}$ for any \mathcal{T} . Thus, $MDD(\mathcal{T}_{\text{Strl}}) \leq \frac{\mu(\mathcal{T}_{\text{Strl}})}{\mu(\mathcal{T})} MDD(\mathcal{T})$ for any \mathcal{T} . Choosing $\mathcal{T} = \mathcal{T}_\mu$ and noting that $\mu(\mathcal{T}_{\text{Strl}}) \leq \mu(\mathcal{T}_\mu)$, we obtain the desired result. ■

Since the cost (in terms of the cumulative return) of entering and exiting a trade is $-(\hat{f}_S + \hat{f}_B)$, no segment of the optimal trading strategy \mathcal{T}_μ should lose more than this in return.

Lemma 3.2 For any $i < j$, $C_{\mathcal{T}_\mu}[j] - C_{\mathcal{T}_\mu}[i] \geq -(\hat{f}_S + \hat{f}_B)$.

Proof: Suppose, for contradiction, that for some $i < j$, $C_{\mathcal{T}_\mu}[j] - C_{\mathcal{T}_\mu}[i] < -(\hat{f}_S + \hat{f}_B)$. By setting $\mathcal{T}_\mu[i+1], \dots, \mathcal{T}_\mu[j]$ to be all equal to 0, it is easy to verify that the cumulative return of the strategy must increase, which contradicts the optimality of \mathcal{T}_μ . ■

For technical convenience, we will assume that the transactions cost when entering a trade is assigned to the time period prior to the entry, and the transactions cost when exiting a trade is assigned to the time period after the trade. Note that just prior to entering, the position is 0 and so it will now have a return of $-\hat{f}_S$, and just after exiting, the position is 0, and will now have a return of $-\hat{f}_B$.

Let $f_{sp} = \hat{f}_S + \hat{f}_B$. From Lemma 3.2, no segment of the optimal strategy can lose more than f_{sp} , and so this immediately gives an upper bound on $MDD(\mathcal{T}_\mu)$. For the trivial strategy that makes no trades, the MDD is 0. If a strategy makes exactly one trade, then there is a drawdown of at least \hat{f}_S at the beginning, and of at least \hat{f}_B at the end. If at least two trades are made, then there is a drawdown of at least f_{sp} between the exit of one trade and the entry of another, and since the drawdown cannot exceed f_{sp} , the MDD must therefore equal f_{sp} . We thus have the following lemma.

Lemma 3.3 (MDD of \mathcal{T}_μ) *If \mathcal{T}_μ makes no trades, $MDD(\mathcal{T}_\mu) = 0$. If \mathcal{T}_μ makes one trade, $\max\{\hat{f}_S, \hat{f}_B\} \leq MDD(\mathcal{T}_\mu) \leq f_{sp}$. If \mathcal{T}_μ makes at least two trades, $MDD(\mathcal{T}_\mu) = f_{sp}$.*

note that if we relax assumption **A1**, then by legging into a trade, it may be possible to decrease the drawdown, in which case Lemma 3.3 would no longer be valid. We are now ready to discuss the $O(n \log n)$ algorithms to obtain Sterling-optimal trading strategies. First we will consider unconstrained Sterling optimal strategies, and then we will require number of trades $\leq K$.

3.1 Unconstrained Sterling-Optimal Strategies

For a degenerate trading system with all returns equal to zero, we define its Sterling ratio as 1. The only trading system with a MDD of 0 is a degenerate trading system, so with this definition, the Sterling ratio is defined for all possible trading systems. The computation of the Sterling-optimal trading system breaks down into three cases, according to the number of trades its makes:

Sterling-optimal that makes zero trades. Sterling Ratio is 1.

Sterling-optimal that makes one trade. The trading strategy contains a single interval of 1's.

Sterling-optimal that makes at least two trades. Any trading system that makes at least two trades has an $MDD \geq f_{sp}$. Since $MDD(\mathcal{T}_\mu) \leq f_{sp}$ (Lemma 3.3), \mathcal{T}_μ has the smallest MDD among all such systems. Since it also has the highest total return, we conclude that if the Sterling-optimal system makes at least two trades, then $\mathcal{T}_{Strl} = \mathcal{T}_\mu$.

The first case is trivially computed. The third case, i.e., the Sterling optimal strategy that makes at least two trades can be computed in linear time using the dynamic programming algorithm to compute \mathcal{T}_μ . If we also compute the Sterling-optimal system that makes exactly one trade, then, we solve our problem by taking the case with the maximum Sterling ratio. We now focus on finding the trading strategy that makes only one trade and has greatest Sterling Ratio among all such strategies.

Let \mathcal{T} be a strategy that makes exactly one trade. The trade interval is the interval of time periods, $[t_i, t_j]$ on which $\mathcal{T} = 1$, i.e., the trade interval is an interval of 1's in the trading strategy. An elementary algorithm that considers all the $O(n^2)$ possible trade intervals, picking the best is a quadratic time algorithm. The remainder of this section is devoted to providing an algorithm which computes such a strategy in $O(n \log n)$ time, which will complete the proof of the first part of Theorem 1.3. In fact the algorithm that we present is a much more general algorithm that computes the single interval that optimizes a general class of optimality criteria. This algorithm will be useful when we discuss the Sharpe-optimal strategy.

Consider a consecutive sequence of time periods $t_i, t_{i+1}, \dots, t_{i+k}$ where $k \geq 1$, with all the excess returns non-negative and the last one positive, i.e., $\hat{s}_i \geq 0, \hat{s}_{i+1} \geq 0, \dots, \hat{s}_{i+k} > 0$.

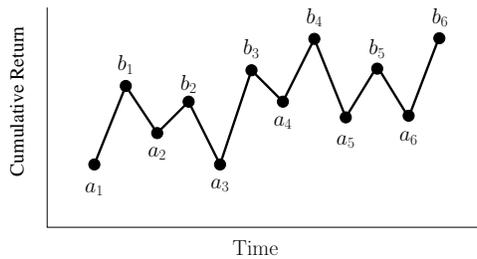
Lemma 3.4 *Either the optimal single trade interval does not intersect these time periods, or an optimal single interval can be chosen to contain this interval.*

Proof: Suppose that $\mathcal{T}[i+j] = 1$ and $\mathcal{T}[i+j+1] = 0$ for some $0 \leq j < k$. Extend the trading interval by setting $\mathcal{T}[i+j+1] = 1, \dots, \mathcal{T}[i+k] = 1$, which adds positive return, without increasing the *MDD*, contradicting the optimality of the original interval. On the other hand, suppose that $\mathcal{T}[i+j] = 1$ and $\mathcal{T}[i+j-1] = 0$ for some $0 < j \leq k$. Once again, by extending the trading interval, setting $\mathcal{T}[i] = 1, \dots, \mathcal{T}[i+j] = 1$, we add non-negative returns, without increasing the *MDD* hence this new interval is at least as good as the previous interval. ■

A similar result holds for a sequence of consecutive negative time periods, $t_i, t_{i+1}, \dots, t_{i+k}$ where $k \geq 1$, with $\hat{s}_i \leq 0, \hat{s}_{i+1} \leq 0, \dots, \hat{s}_{i+k} < 0$. If an optimal trading interval only intersects part of these time periods, this intersection can be removed without decreasing the Sterling ratio. Thus, by Lemma 3.4, any sequence of time periods with all returns non-negative (non-positive) can be condensed into a single time period, $t'_i = t_i + \dots + t_{i+k}$, with $\hat{s}'_i = \hat{s}_i + \dots + \hat{s}_{i+k}$. Further, this operation can be performed in linear time on the entire excess return sequence, so from now on we assume without loss of generality that the excess return sequence consists of alternating time periods of strictly positive and negative excess returns. If $\hat{s}_i < 0$, then t_i cannot be the first 1 of a trade, since by entering one time period later, we exclude only this negative return and do not increase the *MDD*. Similarly, it cannot be the last 1 of a trade.

Lemma 3.5 *The first 1 and the last 1 of the optimal trade interval must occur at time periods t_f and t_l for which $\hat{s}_f > 0$ and $\hat{s}_l > 0$.*

The pictorial illustration of this lemma is given below where we show the cumulative return curve. The time *instants* a_i are the possible entry points, and the time instants b_i are the possible exit points.



Let the alternating sequence of entry and exit points be $\{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$ (a_i are the entry points, and b_i are the exit points). Note that after the preprocessing into alternating intervals, $k \leq \lceil n/2 \rceil$. Notice that without loss of generality, we can throw away the first interval if it has a negative return, as it can never be an entry point, and the last interval if it has a negative return, for a similar reason. The optimal trade interval will be of the form (a_t, b_{t+r}) , $r \geq 0$.

Our algorithm for finding the Sterling-optimal interval will be to consider every possible starting point a_i , and find the Sterling-optimal interval with this point as starting point (i.e. we have to find the end point of this interval). As the algorithm proceeds, we keep track of the best entry point (and its corresponding exit point). The entry points a_t are processed from right to left. After processing a new entry point a_t , we will modify the alternating sequence to facilitate faster processing of the remaining points. More specifically, we will delete the processed entry point and add a weight to the edge between b_{t-1} and b_t to preserve all the necessary information – we cannot simply delete the entry point a_t , since we have to keep track of maximum MDD that occurs in our activity interval. Since between b_{t-1} and a_t we have a drawdown of $b_{t-1} - a_t$, we need to keep this information in an edge weight connecting b_{t-1} to b_t . Please note that at any stage of algorithm edge weight connecting b_{t-1} to b_t will be equal to the MDD of the interval $[b_{t-1}, b_t]$ **and this MDD is realized on prefix of $[b_{t-1}, b_t]$** , i.e. $MDD([b_{t-1}, b_t]) = C(b_{t-1}) - C(x)$, for some $x \in [b_{t-1}, b_t]$ - **Invariant (*)**. We will show this weight attached to b_t in parentheses, $(w_t)b_t$, where the value of w_t appearing in parentheses indicates the weight.

We start our backward scan at the last entry point, a_k , for which there is only one possible interval (a_k, b_k) . We update the weight $w_k \leftarrow b_{k-1} - a_k$, store the current best interval $(a_k, b_k, \text{Strl}_k)$, and delete the possible start point a_k from the sequence to give the processed sequence $\{a_1, b_1, \dots, a_{k-1}, b_{k-1}, (w_k)b_k\}$. Note that $(a_k, b_k, \text{Strl}_k)$ is a one-step trade, but we keep it here for simplicity. We now proceed to a_{k-1} and so on.

In the general case, suppose we have processed (backwards) all the entry points up to (including) the entry point a_{t+1} , and are currently processing entry point a_t . The weighted exit sequence is $\{a_1, b_1, \dots, a_t, b_t, (w_{t+1})b_{t+1}, \dots, (w_{t+m})b_{t+m}\}$. b_t, \dots, b_{t+m} are the possible exit points – note that $t + m$ may not equal k due to possible deletion of points which we discuss below. Assume that $\{b_{t+1} < \dots < b_{t+m}\}$: this is true after we have processed the first start point (since the sequence consists only of one point), and we will maintain this condition by induction. If $b_t < b_{t+1}$, then the entire sequence of exit points is monotonically increasing. On the other hand, if $b_t \geq b_{t+1}$, b_{t+1} need not be considered as an exit point for any optimal interval with entry point a_t or earlier, because by stopping earlier at b_t , we do not decrease the cumulative return, nor increase the MDD . Hence, we can delete the possible exit point b_{t+1} . However, we must now update the weight in $(w_{t+2})b_{t+2}$ to store the new drawdown between b_t and b_{t+2} as follows $w_{t+2} \leftarrow \max\{w_{t+1}, w_{t+2} + b_t - b_{t+1}\}$.

Lemma 3.6 *If $b_t \geq b_{t+1}$, the weighted exit sequence is updated as follows:*

$$a_t, b_t, (w_{t+1})b_{t+1}, \dots, (w_{t+m})b_{t+m} \rightarrow a_t, b_t, (\max\{w_{t+1}, w_{t+2} + b_t - b_{t+1}\})b_{t+2}, \dots, (w_{t+m})b_{t+m}$$

The correctness of the weight updating rule above follows from the Invariant (*). Please note that the transformation above preserves the Invariant (*).

This process is continued until the next exit point after b_t is either above b_t or there are no remaining exit points after b_t . In either event, the new sequence of exit points available for a_t is strictly monotonically increasing (by the induction hypothesis). Observe that any deletion of a possible exit point is a constant time operation. Further, since each deletion drops a point from the set $\{b_1, \dots, b_k\}$, there can be at most $k - 1$ such deletions during the course of the *entire* algorithm. We thus have the following lemma.

Lemma 3.7 *When a_t is processed by the algorithm, the exit points $b_t < b_{t+1} < \dots$ are monotonically increasing. The cost of maintaining this condition for the entire algorithm is $O(k)$ operations.*

When processing a_t , the weighted exit sequence is $\{a_1, b_1, \dots, a_t, b_t, (w_{t+1})b_{t+1}, \dots, (w_{t+m})b_{t+m}\}$. Suppose that $w_{t+2} < w_{t+3} < \dots < w_{t+m}$. Initially this sequence is the empty sequence and so this condition holds, and once again, by induction we will ensure that this condition will always hold. If $w_{t+1} \geq w_{t+2}$, then no optimal interval can have entry point a_t or earlier, and exit at b_{t+1} , because otherwise by exiting at b_{t+2} , since $b_{t+2} > b_{t+1}$ (Lemma 3.4), the *MDD* is not increased, however the total return is increased. Thus if $w_{t+1} \geq w_{t+2}$, we can remove the possible exit point b_{t+1} from the weighted exit sequence and update $w_{t+2} \leftarrow w_{t+1}$. Please note that this transformation also preserves the Invariant (*).

Lemma 3.8 *If $w_{t+1} \geq w_{t+2}$, the weighted exit sequence is updated as follows:*

$$a_t, b_t, (w_{t+1})b_{t+1}, \dots, (w_{t+m})b_{t+m} \rightarrow a_t, b_t, (w_{t+1})b_{t+2}, \dots, (w_{t+m})b_{t+m}$$

We continue removing exit points in this way until either there is only one weight left in the weighted exit sequence, or all the weights are strictly monotonically increasing (by induction).

Suppose that $w_{t+1} \leq f$. In this case, we observe that b_t cannot be the exit of an optimal interval with entry a_{t-r} , where $r \geq 0$. To see this note that if $b_t - a_{t-r} - \hat{f}_S < 0$, then the return of this interval is negative and this interval cannot be an optimal interval. If $b_t - a_{t-r} - \hat{f}_S \geq 0$ then since the interval already has *MDD* of at least f , so by continuing to b_{t+1} , we do not increase the *MDD* but strictly increase the return, hence it cannot be optimal to exit at b_t .

Lemma 3.9 *Let*

$$a_t, b_t, (w_{t+1})b_{t+1}, \dots, (w_{t+m})b_{t+m}$$

be the weighted exit sequence and let $T \in [t, \dots, t+m]$ be such an index that $w_T < f$ and $w_{T+1} \geq f$. Then no Sterling-optimal interval that starts at a_t or earlier can exit at $\{b_t, \dots, b_{T-1}\}$.

Lemma 3.10 *When a_t is processed by the algorithm, the weights w_{t+1} are monotonically increasing. The cost of maintaining this condition for the entire algorithm is $O(k)$ operations.*

We thus assume from now on that when processing entry point a_t , the weighted exit sequence $\{a_1, b_1, \dots, a_t, b_t, (w_{t+1})b_{t+1}, \dots, (w_{t+m})b_{t+m}\}$ with $m \geq 0$ satisfies the conditions of Lemmas 3.7 and 3.10. The first available exit gives the trade interval (a_t, b_T) . If $b_T - a_t - f_{sp} \leq 0$, i.e., if the return is not positive, then this cannot possibly be an optimal interval. Otherwise, the Sterling Ratio is

$$\text{Strl}_t = \frac{b_T - a_t - f_{sp}}{f},$$

where $f_{sp} = \hat{f}_S + \hat{f}_B$ and $f = \max\{\hat{f}_S, \hat{f}_B\}$. Now consider the exit points b_{T+r} , $r > 0$, and suppose that $b_T - w_{T+1} < a_t$. No trade with entry at a_t , exiting at b_{T+r} with $r > 0$ can possibly be optimal, since we could always enter later at $b_T - w_{T+1}$, exit at b_{T+r} , and strictly increase the return without increasing the drawdown. We are thus done with processing the entry point a_t , and we can proceed to a_{t-1} after updating weight w_t and comparing $\frac{b_T - a_t - f_{sp}}{f}$ with the current champion. Similarly, if $b_{\bar{t}} - w_{\bar{t}+1} < a_t$ for some $\bar{t} \in [t, \dots, T-1]$, we are done with the starting point a_t , and we can proceed to a_{t-1} after updating weight w_t . We assume that at any stage of the algorithm we keep the value of $\min_{\bar{t} \in [t, \dots, T-1]} b_{\bar{t}} - w_{\bar{t}+1}$ and thus this check can be done in constant time for any given point a_t . Thus, without loss of generality, we can assume that $b_T - w_{T+1} \geq a_t$ and $b_{\bar{t}} - w_{\bar{t}+1} \geq a_t$ for all $\bar{t} \in [t, \dots, T-1]$. Since $w_{T+1} \geq f$, we conclude that $b_T - a_t \geq f$. A trade, entering at a_t and exiting at b_{T+r} , $r > 0$ has total return $b_{T+r} - a_t - f_{sp}$. The next lemma gives the *MDD*.

Lemma 3.11 *Assume that $b_T - w_{T+1} \geq a_t$ and $b_{\bar{t}} - w_{\bar{t}+1} \geq a_t$ for all $\bar{t} \in [t, \dots, T-1]$. The trade (a_t, b_{T+r}) , $r > 0$, has $MDD = w_{T+r}$.*

Proof: The local maxima of the cumulative return sequence for this trade are $\{0, b_t - a_t - \hat{f}_S, \dots, b_T - a_t - \hat{f}_S, \dots, b_{T+r} - a_t - \hat{f}_S\}$. Since $b_{\bar{t}} - w_{\bar{t}+1} - a_t \geq 0 \forall \bar{t} \in [t, \dots, T-1]$, *MDD* of the interval $[a_t, b_T]$ is equal to \hat{f}_S .

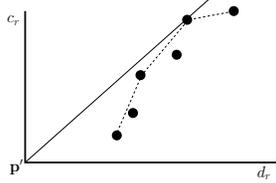
Since $b_T - a_t - \hat{f}_S \geq 0$ and since the sequence of exit points is strictly increasing, $MDD([a_t, b_{T+r}]) = \max(MDD([a_t, b_T]), MDD([b_T, b_{T+r}])) = \max(\hat{f}_S, MDD([b_T, b_{T+r}]), \hat{f}_B)$ where \hat{f}_B is the draw down after the last point b_{T+r} .

Since the drawdown at any time in a trade is given by the the difference between the previous maximum and the current cumulative return, $MDD([b_T, b_{T+r}])$ is at most $\max_{i \in [1, r]} w_{t+i}$. Since the weights are monotonically increasing, we see that this drawdown $\leq w_{t+r}$, which is achieved in the interval (b_{t+r-1}, b_{t+r}) . Since $w_{t+r} \geq f = \max(\hat{f}_S, \hat{f}_B) \forall r > 0$, we conclude $MDD([a_t, b_{T+r}]) = w_{t+r}$. ■

Summary: For entry point a_t , the sequence of exit points b_{T+r} , $r \in [0, m]$ have cumulative returns $c_r = b_{T+r} - a_t - f_{sp}$ and *MDD*'s $d_r = w_{T+r}$ for $r > 0$ and $d_0 = f$. The task is to maximize c_r/d_r with respect to r . The sequences $\{c_r\}$ and $\{d_r\}$ are both strictly increasing. We now describe a general algorithm for performing such a maximization.

3.1.1 Maximizing $\frac{c_r}{d_r}$

On the two dimensional plane, consider the set of points $P'_m = \{(d_r, c_r)\}_{r=0}^m$. Let $\mathbf{p}' = (0, 0)$. Then the slope of the line joining \mathbf{p}' to (d_r, c_r) is exactly the Sterling ratio of the trade (a_t, b_{t+r}) . Thus, finding the optimal trade is equivalent to finding the upper-touching point from \mathbf{p}' to the convex hull of the set of points P'_m (see illustration below). We call \mathbf{p} the source point.



We get the same result if we define $P_m = \{(d_r, b_{t+r})\}_{r=0}^m$, $\mathbf{p} = (0, a_t + f_{sp})$. Given the convex hull, this touching line can be found in time $O(\log A)$ where A is the number of points on the convex hull, [5]. It is easy to see that $A \leq m + 1 \leq \lceil n/2 \rceil$. This algorithm that computes the touching point in $O(\log A)$ requires the ability to efficiently search through the convex hull. We accomplish this by maintaining the convex hull as a doubly linked list where each point in the convex hull maintains $O(\log A)$ pointers to other points in the convex hull. More specifically, point i points forward to points at position 2^j in the convex hull of the points $\{(d_r, c_r)\}_{r=i}^m$, where $j \geq 1$. Each point also maintains backward pointers to any point that it points forward to. At point j , the backward pointers specific to the convex hull starting at point $i < j$ are maintained separately for each i so that constant time access to this set of pointers is possible. An array of an array of pointers suffices. It is clear that the worst case memory requirement is $O(m \log m)$ pointers. We now discuss the main operations we would like to be able to do on our set of points $\{(d_r, c_r)\}$ and the point \mathbf{p} and still be able to compute the upper tangent line efficiently. First we recall that the d_r are monotonically increasing. Assume that each point (d_r, c_r) stores all the necessary forward and backward pointers. We also assume that the point (d_r, c_r) stores the pointer to the point $\text{nxt}(r)$, which is the next point in the convex hull if all the points d_0, \dots, d_{r-1} were removed – note that in this case, d_r becomes leftmost, and so must be in the convex hull. We will see that these assumptions are maintained inductively. Note that the initial convex hull with all the points is given by (d_0, c_0) followed by the points pointed to by $\text{nxt}(0), \text{nxt}(\text{nxt}(0)), \dots$. We would like to perform the following operations on our points and still be able to compute the upper tangent point efficiently:

- (1) Translate \mathbf{p} by some given vector \mathbf{v} .
- (2) Translate all the points in $\{(d_r, c_r)\}$ by some given vector \mathbf{v} .
- (3) Remove the leftmost point (d_0, c_0) .
- (4) Add a new leftmost point (d_{-1}, c_{-1}) .

Lemma 3.12 *Assuming that \mathbf{p} , $\{(d_r, c_r, \text{nxt}_r)\}_{r=1}^m$ are given, all the operations in (1)-(4) above can be accomplished in time $O(\log m)$. Further, in the event that a point is added or deleted, all necessary pointers are maintained.*

Proof: Let $A = O(m)$ be the size of the current convex hull. For (1), we do not change the points at all, we simply compute the new tangent point for $\mathbf{p}' = \mathbf{p} + \mathbf{v}$, which can be accomplished in $O(\log A)$. (2) is equivalent to shifting \mathbf{p} by $-\mathbf{v}$. To prove (3), notice that if we remove (d_0, c_0) , then the new leftmost point becomes (d_1, c_1) and we immediately have the new convex hull $\text{nxt}(1), \text{nxt}(\text{nxt}(1)), \dots$. Thus we can find the new upper tangent point in $O(\log A') = O(\log m)$, where A' is the size of the new convex hull. Further, deleting (d_0, c_0) requires first deleting the backward pointers of the points that it points to $O(\log A)$, and then deleting the point itself, and its forward pointers (it has no backward pointers), $O(\log A)$. To prove (4), note that when we add (d_{-1}, c_{-1}) , $\text{nxt}(-1)$ is exactly the upper tangent point from $\mathbf{p}' = (d_{-1}, c_{-1})$ to the current convex hull. This can be computed in $O(\log A)$. We now need to add all the necessary pointers into the data structure. For each forward pointer we add, we will add the corresponding backward pointer as well. We need a pointer at position 2^j in the convex hull of (d_{-1}, c_{-1}) . But this is exactly the point at position $2^j - 1$ in the convex hull of point $\text{nxt}(-1)$. Since $\text{nxt}(-1)$ maintains a pointer to point 2^j in its convex hull, and this point will have a backward pointer by one step of this same convex hull, we can construct the forward and backward pointer for point 2^j in the convex hull of (d_{-1}, c_{-1}) in constant time, requiring total time $O(\log A') = O(\log m)$ to construct all the new forward and backward pointers, where A' is the size of the new convex hull. We now construct the new upper tangent point from \mathbf{p} to the new convex hull of (d_{-1}, c_{-1}) in $O(\log A')$ time. The entire process is therefore $O(\log m)$. ■

The algorithm that we have just described is a general purpose algorithm for efficiently maintaining the upper tangent point to *any* set of points, as long as only a limited set of operations is allowed on the set of points and the source point. We will now see that these limited operations are all that is needed for maximizing the Sterling ratio.

Suppose that point a_t has been processed in the algorithm – i.e., the upper tangent point (optimal trade with entry at a_t) from $\mathbf{p}_t = (0, a_t + f_{sp})$ to $P_m = \{(d_r, b_{T+r})\}_{r=0}^m$ has been computed. Now consider the addition a_{t-1} to construct the new weighted exit sequence. Delete the leftmost point (d_0, b_T) from the convex hull. Lets consider all possible operations that may take place. There are several possibilities.

- i. $b_{t-1} \geq b_t$. We remove (leftmost) points b_{t+i} , $i \geq 0$, until $b_{t-1} < b_{t+i+1}$, and the new weight w'_{t+i+1} may have increased (Lemma 3.6). Deleting of points $b_t + i$ from the weighted exit sequence doesn't implies changes of the convex hull until $t + i \geq T$. After this point, deleting one point $b_T + i$, $i \geq 0$ from the weighted exit sequence followed by deletion of corresponding

leftmost point of the convex hull. At the very end of the sequence of deletions, we have to update the MDD of point b_{t+i+1} from w_{t+i+1} to w'_{t+i+1} , this can be done by deletion of the point (w_{t+i+1}, b_{t+i+1}) and addition of new point (w'_{t+i+1}, b_{t+i+1}) . The total number of such removals during the entire algorithm is at most $n - 1$. When condition $b_{t-1} < b_t$ is satisfied, proceed to the next stage.

ii. $b_{t-1} < b_t$.

ii.1. $w_{t+1} \leq w_t$. We remove (leftmost) points b_{t+i} , $i \geq 0$, until $w_t < w_{t+i+1}$. Deleting of points $b_t + i$ from the weighted exit sequence doesn't implies changes of the convex hull until $t + i \geq T$. After this point, deleting one point $b_T + i$, $i \geq 0$ from the weighted exit sequence followed by deletion of corresponding leftmost point of the convex hull. By Lemma 3.10, the total number of such removals cannot exceed $n - 1$ over the course of the entire algorithm. When condition $w_t < w_{t+1}$ is satisfied, proceed to the next stage.

ii.2. $b_{t-1} < b_t$ and $w_t < w_{t+1}$.

(a) $f > w_t$. Add to the convex hull point (f, b_T) .

(b) $f < w_t$. Add to the convex hull points (w_t, b_t) and (f, b_{t-1}) .

The new source point is $p_{t-1} = (0, a_{t-1} + f_{sp})$, which just corresponds to a shift of \mathbf{p}_t , and so once the new convex hull for the new weighted exit sequence is computed, an additional $O(\log n)$ operations are needed to find the new upper tangent point.

The total number of removals in the entire algorithm is $O(n)$. For each new entry point, we have at most a constant number of additions, and since the number of entry points is $O(n)$, we see that the total number of additions is $O(n)$. By Lemma 3.12, we have that each operation takes $O(\log n)$ in the worst case, thus the total run time is $O(n \log n)$. Collecting all the results together, we can find the Sterling-optimal strategies making zero, one or more than one trade in $O(n \log n)$ time, completing the proof of the first part of Theorem 1.3.

Note that by only maintaining exit points with weight at most some given constant, MDD_0 , i.e., by truncating some region of the points to the right, this algorithm is easily extended to computing the Sterling-optimal strategy that uses exactly one trade and has an $MDD \leq MDD_0$.

Proposition 3.13 *Given MDD_0 , a Sterling-optimal strategy that uses exactly one trade and has $MDD \leq MDD_0$ can be computed in $O(n \log n)$ time.*

This result will be useful when we consider constrained Sterling-optimal strategies.

3.2 Constrained Sterling-Optimal Strategies

As with the return-optimal strategies, the unconstrained sterling-optimal strategies may make too many trades. Here, we consider the the Sterling-optimal strategy that makes at most K trades,

$\mathcal{T}_{\text{Strl}}^K$. We refer to such strategies as K -Sterling-optimal. First, we present some properties of this strategy, before giving an efficient algorithm to compute it.

A maximal return-optimal strategy \mathcal{T}_μ^* is a return-optimal whose trade intervals cannot be enlarged. Given any return-optimal strategy \mathcal{T}_μ , in one (linear time) scan from left to right, we can enlarge any trade intervals maximally to the right as long as they keep the same return. Similarly, in one backward scan, we can extend all trade intervals maximally to the left. Since \mathcal{T}_μ can be computed in linear time, we conclude that

Lemma 3.14 *A maximal return-optimal strategy \mathcal{T}_μ^* can be computed in linear time.*

If any trade interval of a maximal return-optimal strategy is extended in either direction, then the total return must strictly decrease. In the previous section, we gave a $O(n \log n)$ algorithm for computing the Sterling-optimal strategy with exactly 1 trade. We also saw that if the unconstrained Sterling-optimal strategy contains more than 1 trade, then it is \mathcal{T}_μ^* . Fix K , and let the number of trades that \mathcal{T}_μ^* makes be $K_0 \leq K$. In this case $\mathcal{T}_{\text{Strl}}^K = \mathcal{T}_\mu^*$, and we are done. Thus we only need to consider the case that $1 < K < K_0$. Some important properties of \mathcal{T}_μ^* are summarized below. When it is clear, we also use \mathcal{T}_μ^* to refer to the set of trading intervals $\{I_r\}_{r=1}^{K_0}$. Let $C_i = \sum_{j=1}^i \hat{s}_j$ denote the cumulative return sequence of the excess returns.

Lemma 3.15 *Let \mathcal{T}_μ^* be maximal return-optimal. Let I be an interval $[t_i, t_j]$.*

- i. If $I \in \mathcal{T}_\mu^*$, then, $\sum_{k=i}^j \hat{s}_k - f_{sp} \geq 0$ and $MDD(I) \leq f_{sp}$.*
- ii. Suppose I does not intersect with any interval in \mathcal{T}_μ^* and let the return of I ($\sum_{k=i}^j \hat{s}_k$) be $\mu(I)$. Then, $\mu(I) \leq f_{sp}$. If I is adjacent to some interval of \mathcal{T}_μ^* , then $\mu(I) < 0$. If I is adjacent to two intervals in \mathcal{T}_μ^* , then $\mu(I) < -f_{sp}$.*
- iii. Let $[t_l, t_r]$ and $[t_{l'}, t_{r'}]$ be two consecutive trade intervals in \mathcal{T}_μ^* , $l \leq r < l' \leq r'$. Then, $C_r - C_{l'} > f_{sp}$, and for all $r < q < l'$, $C_{l'} < C_q < C_r$.*

Let $\{(a_i, b_i)\}$ denote the local minima and maxima of $\{C_i\}$, as in the previous section. Any trade of \mathcal{T}_μ^* or $\mathcal{T}_{\text{Strl}}^K$ must enter (exit) at a local minimum (maximum). Further, the entry (exit) point must be a minimum (maximum) in the trade, otherwise we can shrink the trade, strictly increasing the return without increasing the MDD .

Lemma 3.16 *Let $I = [t_l, t_r]$ be a trade interval of \mathcal{T}_μ^* or $\mathcal{T}_{\text{Strl}}^K$. Then C_l is a local minimum, C_r is a local maximum, and for any k , with $l \leq k \leq r$, $C_l \leq C_k \leq C_r$.*

We now give an important inclusion property of the $\mathcal{T}_{\text{Strl}}^K$.

Proposition 3.17 *Let \mathcal{T}_μ^* be a maximal return-optimal trading strategy. There exists a K -Sterling-optimal strategy $\mathcal{T}_{\text{Strl}}^K$, $K > 1$, with the following property: if $I = [t_l, t_r]$ is any trading interval in $\mathcal{T}_{\text{Strl}}^K$, then a prefix of I and a suffix of I are trades in the maximal return-optimal strategy \mathcal{T}_μ^* .*

Proof: First we show that for every trading interval $I^* = [t_a, t_b]$ in \mathcal{T}_μ^* with $I \cap I^* \neq \emptyset$, one can pick $\mathcal{T}_{\text{Strl}}^K$ such that $I^* \subseteq I$. Suppose to the contrary, that for some I^* , either $t_a < t_l$ and $t_b \geq t_l$ or $t_a \leq t_r$ and $t_b > t_r$. We will extend I without decreasing the Sterling ratio of $\mathcal{T}_{\text{Strl}}^K$ so that $I^* \subseteq I$. Suppose $t_a < t_l$ and $t_b \geq t_l$ (a similar argument holds for $t_a \leq t_r$ and $t_b > t_r$). There are two cases:

- i. I^* does not intersect any other interval of $\mathcal{T}_{\text{Strl}}^K$: Applying Lemma 3.16 to I^* , we have: $C_a \leq C_l$. Thus by extending I to $[t_a, t_r]$, the return of the interval cannot decrease. Since $MDD(I^*) \leq f_{sp}$, this extension cannot increase the $MDD(\mathcal{T}_{\text{Strl}}^K)$, since we already have that $MDD(\mathcal{T}_{\text{Strl}}^K) \geq f_{sp}$.
- ii. I^* intersects with the previous trading interval of strategy $\mathcal{T}_{\text{Strl}}^K$: $\exists I' = [t_{l'}, t_{r'}] \in \mathcal{T}_{\text{Strl}}^K$ such that $t_a \leq t_{r'} < t_l$. Since $[t_{r'+1}, t_{l-1}]$ is a subinterval of I^* , $\sum_{j=r'+1}^{l-1} \hat{s}_j \geq -f_{sp}$ (Lemma 3.15). If we merge I and I' by adding the interval $[t_{r'+1}, t_{l-1}]$ into $\mathcal{T}_{\text{Strl}}^K$, we save on the transaction cost of f_{sp} , and so the total return will not decrease. We show that the MDD has not increased. Since $C_{r'}$ is a maximum in $[t_{l'}, t_{r'}]$, the drawdown for all points in $[t_{r'+1}, t_l]$ is at most f_{sp} . Since C_l is a minimum in $[t_l, t_r]$, we conclude that the drawdown for any point in $[t_l, t_r]$ is at most $\max\{f_{sp}, MDD(I)\}$. Since $MDD(\mathcal{T}_{\text{Strl}}^K) \geq f_{sp}$, we conclude that this merger does not increase the MDD .

Note that $\mu(I) \geq 0$ otherwise we improve the return of $\mathcal{T}_{\text{Strl}}^K$ by removing I , without increasing the MDD , and so $\mathcal{T}_{\text{Strl}}^K$ cannot possibly be optimal. Thus, without loss of generality, we assume that the return of I is positive. Suppose that $I \cap \mathcal{T}_\mu^* = \emptyset$. Then, by adding I to \mathcal{T}_μ^* , we strictly increase the return, a contradiction on the optimality of \mathcal{T}_μ^* . Thus, every interval of $\mathcal{T}_{\text{Strl}}^K$ contains at least one interval of \mathcal{T}_μ^* . Now consider the maximal prefix P_{max} of I that does not overlap with any interval of \mathcal{T}_μ^* . Since we know that I contains some interval of \mathcal{T}_μ^* , we conclude that this maximal prefix must be adjacent to some interval of \mathcal{T}_μ^* . By Lemma 3.15, this interval has strictly negative return, so removing it from I strictly increase the return of $\mathcal{T}_{\text{Strl}}^K$, without increasing its MDD . This contradicts the optimality of $\mathcal{T}_{\text{Strl}}^K$, thus, P_{max} must be empty. Similarly, the maximal suffix of I that is non-intersecting with \mathcal{T}_μ^* must be empty, concluding the proof. \blacksquare

As a result of Proposition 3.17, we assume from now on that every interval of the sterling optimal strategy $\mathcal{T}_{\text{Strl}}^K$ is prefixed and suffixed by (not necessarily distinct) intervals from a maximal return-optimal strategy that makes K_0 trades.

Lemma 3.18 *If $1 < K \leq K_0$ then $\mathcal{T}_{\text{Strl}}^K$ can be chosen to make exactly K trades.*

Proof: If $K = K_0$, then \mathcal{T}_μ^* itself is K-Sterling-optimal. If $K < K_0$, we show that if the number of trades made is less than K , we can always add one more interval without decreasing the Sterling ratio of the strategy. First, note that $\mathcal{T}_{\text{Strl}}^K$ cannot contain all the intervals of \mathcal{T}_μ^* , as otherwise (by the pigeonhole principle) at least one interval $I = [t_l, t_r]$ of $\mathcal{T}_{\text{Strl}}^K$ contains two consecutive intervals $I_1 = [t_{l_1}, t_{r_1}]$ and $I_2 = [t_{l_2}, t_{r_2}]$ of \mathcal{T}_μ^* . The region between these two intervals has return less than $-f_{sp}$ (Lemma 3.15), so breaking up I into the two intervals $[t_l, t_{r_1}]$ and $[t_{l_2}, t_r]$ will strictly increase the return, without increasing the MDD , contradicting the optimality of $\mathcal{T}_{\text{Strl}}^K$. If $\mathcal{T}_{\text{Strl}}^K$ does not contain some interval of \mathcal{T}_μ^* , then by adding this interval, we do not decrease the return or the MDD (Lemma 3.15), since the MDD is already $\geq f_{sp}$. ■

Lemmas 3.17 and 3.18 indicate how $\mathcal{T}_{\text{Strl}}^K$ can be constructed: start with all the intervals of a maximal return-optimal strategy \mathcal{T}_μ^* and then merge some neighbouring intervals, keeping the merging sequence that gave the best strategy. The number of possible merging sequences is exponential, however, we will now show that an efficient greedy merging algorithm gives the correct result.

Given two consecutive non-adjacent intervals $I_1 = [t_{l_1}, t_{r_1}]$, $I_2 = [t_{l_2}, t_{r_2}]$, where I_1 precedes I_2 , define the *bridge* $B(I_1, I_2) = [t_{r_1}, t_{l_2}]$ to be interval connecting I_1 with I_2 . If I_1 and I_2 are intervals in a maximal return optimal strategy, then by Lemma 3.15, the MDD of the bridge is $C_{r_1} - C_{l_2}$. Since C_{r_1} is a maximum over the interval $[t_{l_1}, t_{l_2}]$, and C_{l_2} is a minimum over the interval $[t_{r_1}, t_{r_2}]$, we have that the MDD of the union of these three intervals, $[t_{l_1}, t_{r_2}]$ is given by $\max\{MDD(I_1), C_{r_1} - C_{l_2}, MDD(I_2)\}$.

For every bridge $B(I_1, I_2)$, define the *closure* $Cl(B(I_1, I_2))$ to be the *smallest* interval $J = [t_l, t_r]$, in the return sequence, satisfying the following three properties.

Cl_1 . $C_l \leq C_m \leq C_r$ for $l \leq m \leq r$, i.e., C_l is a minimum and C_r is a maximum in $[t_l, t_r]$.

Cl_2 . $I_1, I_2 \subset J$, i.e., J contains both I_1 and I_2 .

Cl_3 . J is prefixed and suffixed by intervals from a maximal return-optimal strategy \mathcal{T}_μ^* .

Note that a bridge may not have closure. For example, if the two last intervals I_{l-1}, I_l in \mathcal{T}_μ^* are such that the end point I_l is below the end point of I_{l-1} , then $B(I_{l-1}, I_l)$ doesn't have a closure. The next lemma shows that if the closure J for a bridge exists, then not only is it unique, but any other interval satisfying $Cl_1 - Cl_3$ contains J .

Lemma 3.19 *For two intervals I_1, I_2 , if $Cl(B(I_1, I_2))$ exists, then it is unique. Moreover, for any other interval I satisfying $Cl_1 - Cl_3$, $Cl(B(I_1, I_2)) \subseteq I$.*

Proof: Let $J_1 = [t_{l_1}, t_{r_1}]$ and $J_2 = [t_{l_2}, t_{r_2}]$ satisfy $Cl_1 - Cl_3$. Without loss of generality, assume that $t_{l_1} \leq t_{l_2} < t_{r_1} \leq t_{r_2}$. By construction, $J_1 \cap J_2 = [t_{l_2}, t_{r_1}]$ satisfies $Cl_1 - Cl_3$. Now let $Cl(B(I_1, I_2))$ be the intersection of all intervals that satisfy $Cl_1 - Cl_3$, concluding the proof. ■

Suppose that bridge B and B' are bridges in \mathcal{T}_μ^* and that $\mathcal{Cl}(B)$ contains B' . Then $\mathcal{Cl}(B)$ satisfies $\mathcal{Cl}_1 - \mathcal{Cl}_3$ with respect to B' and hence $\mathcal{Cl}(B)$ also contains $\mathcal{Cl}(B')$.

Lemma 3.20 *Let B and B' be bridges in \mathcal{T}_μ^* . If $B' \subset \mathcal{Cl}(B)$, then $\mathcal{Cl}(B') \subset \mathcal{Cl}(B)$.*

Any interval in $\mathcal{T}_{\text{Strl}}^K$ containing bridge B satisfies properties $\mathcal{Cl}_1 - \mathcal{Cl}_3$ (Lemma 3.16 & Proposition 3.17), immediately yielding the following proposition.

Proposition 3.21 *Let $I \in \mathcal{T}_{\text{Strl}}^K$ and let B be a bridge in \mathcal{T}_μ^* .*

- i. If $B \subset I$, then $\mathcal{Cl}(B) \subset I$.*
- ii. If B does not have a closure, then no K -Sterling-optimal strategy can contain B .*
- iii. A K -Sterling-optimal strategy with more than one trading interval and no bridges of \mathcal{T}_μ^* has $MDD = f_{sp}$. If it contains one or more bridges B_i of \mathcal{T}_μ^* , then $MDD = \max_i MDD(\mathcal{Cl}(B_i))$.*
- iv. The MDD of a K -Sterling-optimal strategy with more than one trading interval can be one of at most $T + 1$ possible values where T is the number of bridges between the intervals of \mathcal{T}_μ^* .*

Proof: (i) and (ii) are immediate. (iv) follows from (iii), thus we only need to prove (iii). Let $I \in \mathcal{T}_{\text{Strl}}^K$ contain the consecutive bridges B_1, \dots, B_K , and hence their closures. From (i), it is clear that $MDD(I) \geq \max_i MDD(\mathcal{Cl}(B_i))$. It is also clear that $I = \cup_i^K \mathcal{Cl}(B_i)$. We show, by strong induction on K , a more general statement than we need: suppose that $I = \cup_i^K \mathcal{Cl}(B_i)$, then $MDD(I) \leq \max_i MDD(\mathcal{Cl}(B_i))$. If $K = 1$ then $I = \mathcal{Cl}(B_1)$ and the result is trivial; suppose it is true for up to $K - 1$ consecutive bridges, $K > 1$, and suppose that I is the union of K closures of consecutive bridges. Consider the first closure $\mathcal{Cl}(B_1)$. Let $I = [t_l, t_r]$ and $\mathcal{Cl}(B_1) = [t_l, t_{r'}]$, $t_{r'} \leq t_r$. By definition of $\mathcal{Cl}(B_1)$, $C_{r'}$ is a maximum over $[t_l, t_{r'}]$. Thus, $MDD(I) = \max\{MDD(\mathcal{Cl}(B_1)), MDD([t_{r'}, t_r])\}$. If $r = r'$, then $I = \mathcal{Cl}(B_1)$ and we are done. If $r < r'$, then $t_{r'+1}$ is the beginning of some bridge B_κ . Let $I' = \cup_{i=\kappa}^K \mathcal{Cl}(B_i)$. Then, $[t_{r'}, t_r] \subseteq I'$ and so $MDD([t_{r'}, t_r]) \leq MDD(I')$. But I' is the union of at most $K - 1$ closures, so by the induction hypothesis, $MDD(I') \leq \max_{i \geq \kappa} MDD(\mathcal{Cl}(B_i))$, concluding the proof. ■

We will distinguish between four types of bridges. Let $I_1 = [t_{l_1}, t_{r_1}]$, $I_2 = [t_{l_2}, t_{r_2}]$ be consecutive intervals in \mathcal{T}_μ^* . The bridge $B = B(I_1, I_2)$ can be one of four types:

- regular.* $C_{l_1} \leq C_{l_2}$ and $C_{r_1} \leq C_{r_2}$, i.e., $\mathcal{Cl}(B) = [l_1, r_2]$.
- right irregular.* $C_{l_1} \leq C_{l_2}$ and $C_{r_1} > C_{r_2}$, i.e., $\mathcal{Cl}(B)$ contains the next bridge.
- left irregular.* $C_{l_1} > C_{l_2}$ and $C_{r_1} \leq C_{r_2}$, i.e., $\mathcal{Cl}(B)$ contains the previous bridge.
- irregular.* $C_{r_1} > C_{r_2}$ and $C_{l_1} > C_{l_2}$, i.e., $\mathcal{Cl}(B)$ contains both the next and previous bridges.

We define the *weight* of the bridge $W(B(I_1, I_2))$ as follows:

$$W(B(I_1, I_2)) = \begin{cases} C_{r_1} - C_{l_2} & \text{if } B(I_1, I_2) \text{ is regular,} \\ C_{r_1} - C_{l_2} & \text{if } B(I_1, I_2) \text{ is left irregular and the previous bridge is right irregular.} \\ C_{r_1} - C_{l_2} & \text{if } B(I_1, I_2) \text{ is left irregular and the previous bridge is irregular.} \\ +\infty & \text{otherwise.} \end{cases}$$

The general idea behind our algorithm is to start with a maximal return-optimal strategy and greedily merge pairs of intervals or pair of bridges according to the bridge weight, keeping track of the best K intervals each time. When no more merging can occur, because we are down to K intervals or all the bridge weights are ∞ , we return the best K intervals we have seen so far. More precisely, let $\mathcal{T}_\mu^* = \{I_1, \dots, I_{K_0}\}$ be a maximal return-optimal trading strategy making K_0 trades. We denote this pool of trade intervals by P_0 , the *base pool*. From pool P_i , we obtain pool P_{i+1} by a single merge according to the following rule. Let $B = B(I_1, I_2)$ be the bridge with smallest weight. If $B = \infty$, stop (pool P_{i+1} does not exist). Otherwise, there are two cases.

- i. *Regular merge*: if B is regular, merge B with I_1 and I_2 to get a larger interval $I_{new} = [t_{l_1}, t_{r_2}]$.

We now update the status (type and weight) of any neighboring bridges as follows:

- Previous bridge changes from right-irregular to regular (update type and weight).
- Previous bridge B' changes irregular to left-irregular (update type). If the bridge previous to B' is right-irregular or irregular then update weight.
- Next bridge changes from irregular to right-irregular (update type).
- Next bridge changes from left-irregular to regular (update type and weight).

- ii. *Irregular merge*: if B is left irregular, denoting the previous bridge B^* , merge the two bridges and the interval between them into one bigger bridge $B_{new} = B^* \cup I_1 \cup B$. The status of bridges other than B have not changed. The status and weight of B may need updating.

Intervals are formed only by regular merges, so it is easy to see that intervals resulting from this merging procedure begin at a minimum and end at a maximum. New bridges are formed by irregular merges, and the resulting bridge must begin at a maximum and end at a minimum. The only bridge weights that could change are those that had weights of ∞ . In such an event the weight will drop, but not below the weight of the original bridge used in the merge that led to the update.

Lemma 3.22 *Let bridge B with weight w be involved in a merge, and suppose that the weight of bridge B' is updated in the process from ∞ to u . Then, $w < u$.*

Proof: There are two cases:

- i. The bridge involved in the merge was regular, i.e., two consecutive intervals $I_1 = [t_{l_1}, t_{r_1}]$ and $I_2 = [t_{l_2}, t_{r_2}]$ are merged with their bridge $B_{12} = B(I_1, I_2)$ with $W(B_{12}) = w$. Let the preceding interval be $I_0 = [t_{l_0}, t_{r_0}]$ and the following interval be $I_3 = [t_{l_3}, t_{r_3}]$, and let the preceding and following bridges be B_{01} and B_{23} respectively. If B_{01} obtained finite weight, it cannot be left irregular, as it would remain left irregular after the merge, and hence its weight would still be ∞ . Thus, we need only consider B_{01} right irregular or irregular (i.e., $C_{r_0} > C_{r_1}$). Its weight becomes $u = C_{r_0} - C_{l_1} > C_{r_1} - C_{l_1}$. Since B_{12} is regular, $w = C_{r_1} - C_{l_2} < C_{r_1} - C_{l_1}$ and so $w < u$. If B_{23} obtained finite weight, then it could not be right regular or irregular as it could not become regular or left irregular after the merge. Thus, we need only consider B_{23} left irregular ($C_{l_2} > C_{l_3}$). Its weight becomes $u = C_{r_2} - C_{l_3} > C_{r_2} - C_{l_2}$. Since B_{12} is regular, $C_{r_1} \leq C_{r_2}$, and so $u > C_{r_1} - C_{l_2} = w$.
- ii. The bridge involved in the merge was left-irregular, i.e., $B_{12} = [t_{r_1}, t_{l_2}]$ is left irregular, and $B_{01} = [t_{r_0}, t_{l_1}]$ is either right-irregular or irregular (in both cases, $C_{r_0} > C_{r_1}$). Let $w = C_{r_1} - C_{l_2}$ be the weight of B_{12} . The merged bridge is $B = B_{01}I_1B_{12}$. If B has finite weight (i.e. it is either regular or left-irregular), then its new weight is $u = C_{r_0} - C_{l_2} > C_{r_1} - C_{l_2} = w$. If B is left-irregular or irregular, then it does not change the weight of any other bridge. If, on the other hand, B became right-irregular or irregular, then it could affect the weight of the following bridge B_{23} , if B_{23} was left-irregular ($C_{l_2} > C_{l_3}$). In this case, the weight of $B_{23} = [t_{r_2}, t_{l_3}]$ becomes $v = C_{r_2} - C_{l_3} > C_{r_2} - C_{l_2}$. But since B_{12} was left-irregular, $C_{r_2} \geq C_{r_1}$, and so $v > C_{r_1} - C_{l_2} = w$.

■

The next lemma shows that if all the bridge weights become ∞ , any further merged pairs of intervals can never be part of a K -Sterling-optimal strategy.

Lemma 3.23 *If all the bridge weights in a pool of intervals are ∞ , then any further merged pairs of intervals from this pool can never be part of a K -Sterling-optimal strategy.*

Proof: (Lemma 3.23). Let P_r be pool obtained from P_0 by some sequence of merging intervals with bridges of finite weight, and suppose that all the bridges in P_r have infinite weight. In particular, this means that none of the bridges in P_r are regular. Denote the bridges by B_1, \dots, B_m , and consider bridge B_k . If B_k is right irregular or irregular, then all following bridges are either right irregular or irregular since all bridges have finite weight. If a trading interval contains B_k , it must contain B_{k+1} (since B_k is right irregular or irregular), and so by induction, it must contain all the following bridges (and their closures). But, the last bridge does not have a closure (as it is right

irregular or irregular), a contradiction. If on the other hand, B_k is left irregular, then all preceding bridges are left irregular as all bridges have infinite weight. If a trading interval contains B_k , it must contain B_{k-1} (since B_k is left irregular), and so by induction, it must contain all the preceding bridges (and their closures). But, the first bridge does not have a closure (as it is left irregular), a contradiction. We conclude that B_k cannot be in any trading interval. ■

Each merge decreases the number of intervals and number of bridges by one. If we merge down to pool P_{K_0-K} , we are left with exactly K intervals. We will show that $\mathcal{T}_{\text{Strl}}^K$ can be chosen to be the best K trades (with respect to total return) in one of these pools. Specifically, define \mathcal{T}_j^K to be the K intervals in P_j with the highest total return. We say that a strategy is *coarser* than pool P_i if the strategy can be obtained by a sequence of merges of some (or all) of the intervals in P_i . Clearly, $\forall i$, P_{i+1} is coarser than P_i , as P_{i+1} is obtained from P_i after a single merge. Note that for a strategy to be coarser than P_i , it need not contain every trade in P_i , however if it contains part of any trade in P_i , then it contains the entire trade. Next, we show that after a merge, the *MDD* of the remaining intervals is equal to the weight of the bridge involved in the merging.

Lemma 3.24 *If pool P_i , $i \geq 1$, was obtained from P_{i-1} by a merge involving a bridge of weight w , then the *MDD* of any interval in P_i is at most w . If the merge created a new interval (i.e., the bridge was regular), then the *MDD* of the new interval is equal to w .*

Proof: In pool P_0 , since any bridge is adjacent to two intervals of \mathcal{T}_μ^* , its weight is at least f_{sp} (Lemma 3.15). Consider sequence of pools P_0, P_1, \dots, P_r , where bridge B_i with weight $W(B_i)$ was the minimum weight bridge involved in the merge that resulted in pool P_i from from P_{i-1} . By Lemma 3.22 bridges weights are non-decreasing, i.e., $W(B_i) \leq W(B_{i+1})$.

We now use induction on the index i . For $i = 1$, from Lemma 3.15, every interval in P_0 has *MDD* at most f_{sp} . If P_1 was obtained from P_0 by an irregular merge, then all intervals of P_1 are intervals of P_0 , with *MDD* at most f_{sp} . Since $W(B_1) \geq f_{sp}$, the claim holds. If the merge was regular, then the *MDD* is $W(B_1) \geq f_{sp}$ and the *MDD* of all other intervals is at most f_{sp} . Thus, the claim holds for P_1 .

Suppose the claim holds for all $j < i$ and consider pool P_i which was obtained from P_{i-1} using a merge involving B_i . By the induction hypethesis, the *MDD* of any interval from P_{i-1} is at most $W(B_{i-1}) \leq W(B_i)$. If P_i that was obtained by an irregular merge, every interval of P_i is an interval of P_{i-1} and thus has *MDD* at most $W(B_{i-1}) \leq W(B_i)$. Suppose that P_i was obtained by a regular merge – all intervals except the merged interval are intervals of P_{i-1} . Consider the *MDD* of the new interval, which is obtained by the regular merge $I_1 \cup B_i \cup I_2$. Since new intervals are created only through regular merges, it is easy to see by induction that property \mathcal{Cl}_1 holds for all the intervals in P_{i-1} , in particular it holds for I_1 and I_2 . Since B_i was regular, the *MDD* of the new

interval is $\max(MDD(I_1), W(B_i), MDD(I_2))$. By the induction hypothesis, $MDD(I_1) \leq W(B_{i-1})$ and $MDD(I_2) \leq W(B_{i-1})$, thus, $\max(MDD(I_1), W(B_i), MDD(I_2)) = W(B_i)$. ■

First, we show that if a K -Sterling-optimal strategy makes K trades, all of which are contained in intervals of one of the pools P_i , then a K -Sterling-optimal strategy exists which is composed of the K intervals with highest return in some pool P_j with $j \leq i$.

Lemma 3.25 *If K subintervals of the intervals of pool P_i are a K -Sterling-optimal strategy, then for some $j \leq i$, the K intervals with highest return of pool P_j are a K -Sterling-optimal strategy.*

Proof: If $P_i = P_0$, then the claim is trivial. Suppose that $i > 0$, and let $\mathcal{T} = \{I_1, \dots, I_K\}$ be the K -Sterling-optimal strategy whose trades are all subintervals of intervals in P_i . Consider the set \mathcal{B} of all bridges in \mathcal{T}_μ^* that are contained in \mathcal{T} , $\mathcal{B} = \{B_i\}_{i=1}^r$. We can assume that \mathcal{B} is not empty because if it were, then \mathcal{T} is composed of intervals in \mathcal{T}_μ^* , in which case the top K intervals (with respect to return) in \mathcal{T}_μ^* are clearly optimal. Since P_i contains all the intervals in \mathcal{T} , P_i contains all the bridges in \mathcal{B} . Thus, there must exist $j \leq i$ such that P_j contains all the bridges in \mathcal{B} and no pool P_k , with $k < j$ has this property, i.e., P_j was obtained from the previous pool by a regular merge involving a bridge B^* which must contain some bridge $B_l \in \mathcal{B}$. Let I be the interval in \mathcal{T} that contains B_l . Then, I must contain the whole bridge B^* , since if B^* is the result of irregular merges, one of which involved bridge B_l , then $B^* \subset Cl(B_l)$, and $Cl(B_l) \subseteq I$ (Proposition 3.21). Since $B \subset I$, $MDD(\mathcal{T}) \geq MDD(I) \geq W(B^*)$. By Lemma 3.24, since B^* was the last bridge involved in a merge, the MDD of every interval in P_j is at most $W(B^*)$. Since every interval of \mathcal{T} is a subinterval of some interval in P_j , we conclude that \mathcal{T} is contained in at most K intervals of P_j . Let \mathcal{T}_j^K be the top K intervals in P_j . Then, the return of \mathcal{T} is at most the return of \mathcal{T}_j^K . Further, $MDD(\mathcal{T}_j^K) \leq W(B^*) \leq MDD(\mathcal{T})$, and so $\text{Strl}(\mathcal{T}_j^K) \geq \text{Strl}(\mathcal{T})$, concluding the proof. ■

We are now ready to prove the main result, which will lead to the greedy algorithm for constructing a K -Sterling optimal strategy.

Theorem 3.26 *Let j^* be such that $\text{Strl}(\mathcal{T}_{j^*}^K) \geq \text{Strl}(\mathcal{T}_j^K)$, $\forall j$. Then $\mathcal{T}_{j^*}^K$ is K -Sterling optimal.*

Proof: Let S_0^K be a K -Sterling-optimal strategy that makes K trades – by Lemma 3.18, such a strategy must exist. If S_0^K has the same Sterling ratio as the trading strategy composed of the K most profitable trades in P_0 , then we are done. If not, then we know from Proposition 3.17 that S_0^K is coarser than P_0 . We prove the following statement for all $k \geq 1$

$Q(k)$: Suppose there exists a K -Sterling-optimal strategy S_{k-1}^K that makes K trades and is coarser than P_{k-1} . Then either S_{k-1}^K is composed of K intervals of P_k , or there exists a K -Sterling-optimal strategy S_k^K that makes K trades and is coarser than P_k .

We know that $Q(1)$ is true. Suppose that $Q(k)$ is true for all $k \geq 1$, we then prove the proposition as follows. By an easy induction, we have that if none of the S_{j-1}^K are composed of K intervals in P_j for all $j \leq m$, then there is a K -Sterling-optimal strategy S_m^K making exactly K trades that is coarser than P_m . Suppose that we can construct a total of $\kappa + 1$ pools, P_i for $0 \leq i \leq \kappa \leq K_0 - K$. If $\kappa < K_0 - K$ then all the bridge weights in P_κ are infinite. If $\kappa = K_0 - K$, then any further merging leads to fewer than K intervals. In both cases, there cannot exist a K -Sterling-optimal strategy that is coarser than P_κ . Therefore, for some $j^* \leq \kappa$, the K -Sterling-optimal strategy $S_{j^*-1}^K$ is composed of K intervals of P_{j^*} . By Lemma 3.25, there is a K -Sterling-optimal strategy T_{Strl}^K that is composed of the top K intervals of some pool P_l , where $l \leq j^*$.

What remains is to show that $Q(k)$ is true for all $k \geq 1$. Suppose that S_{k-1}^K is coarser than P_{k-1} and is not composed of K intervals in P_k . We show that there exists S_k^K that is coarser than P_k . Since S_{k-1}^K is coarser than P_{k-1} , it contains at least one bridge B in P_{k-1} with finite weight (because if it contains an infinite weight bridge, then it either contains the preceding or following bridge; this argument continues analogously to the proof of Lemma 3.23 until we include a bridge of finite weight). Let I be the interval of S_{k-1}^K that contains B , and let I_l and I_r be intervals in P_{k-1} (which are subintervals of I) connected by B . Let B^* be the bridge in P_{k-1} with minimum weight that was involved in the merge to get P_k from P_{k-1} , and let I_l^* and I_r^* be the intervals in P_{k-1} connected by B^* . If $B^* = B$ then S_{k-1}^K is also coarser than P_k and we are done, so suppose $B^* \neq B$. There are two possibilities:

- (i) *B^* is a regular bridge.* If S_{k-1}^K does not contain I_l^* or I_r^* , then $S_{k-1}^K \cap (I_l^* \cup B^* \cup I_r^*) = \emptyset$ and thus S_{k-1}^K itself is coarser than P_k , and can be chosen as S_k^K . Suppose that S_{k-1}^K contains I_l^* and not I_r^* (similar argument if it contains I_r^* and not I_l^*). Thus some interval $I' \in S_{k-1}^K$ has as a suffix I_l^* . Suppose we construct S_k^K by replacing interval I' by interval $I' \cup B^* \cup I_r^*$. S_k^K is then coarser than P_k . Since B^* is regular, the return of $I' \cup B^* \cup I_r^*$ is at least as big as the return of I' . I_r^* is either an interval of P_0 or was obtained by merging some intervals of P_0 through bridges with weight at most $W(B^*)$ (Lemma 3.24), and so $MDD(I_r^*) \leq W(B^*)$. Since the maximum cumulative return for I' is attained at its right endpoint (Lemma 3.16) and the left endpoint of I_r^* is a minimum in I_r^* , we have that $MDD(I' \cup B^* \cup I_r^*) = \max\{MDD(I'), W(B^*), MDD(I_r^*)\} = \max\{MDD(I'), W(B^*)\}$. Since $W(B^*) \leq W(B)$, we conclude that $MDD(S_k^K) \leq MDD(S_{k-1}^K)$, and thus $\text{Strl}(S_k^K) \geq \text{Strl}(S_{k-1}^K)$, which means that S_k^K is also K -Sterling-Optimal. Finally, suppose that S_{k-1}^K contains both I_l^* and I_r^* , and consider the strategy S_k^K obtained from S_{k-1}^K by removing bridge B and adding bridge B^* . $\mu(S_k^K) = \mu(S_{k-1}^K) + W(B) - W(B^*) \geq \mu(S_{k-1}^K)$. Since $W(B) \geq W(B^*)$, the MDD cannot have increased, and so S_k^K is K -Sterling-Optimal and coarser than P_k .

- (ii) *B^* is an irregular bridge.* Since $B^* = B(I_l^*, I_r^*)$ has finite weight, we can conclude that B^* is

left-irregular and the previous bridge $B_- = B(I_{l-1}^*, I_l^*)$ is right-irregular or irregular. Since S_{k-1}^K does not contain B^* , by Lemma 3.16, there are two possibilities: S_{k-1}^K does not contain I_l^* , in which case it also does not contain bridge B_- and so B_- and B^* can be merged into one bridge without influencing S_{k-1}^K , i.e., S_{k-1}^K is also more coarse than P_k ; or, S_{k-1}^K contains I_l^* as one of its intervals. In this case, since B^* is left-irregular, $\mu(I_l^*) < W(B^*) \leq W(B)$, and so by dropping I_l^* from S_{k-1}^K and breaking I into two subintervals by removing B from I results in a profit increase of $W(B) - \mu(I_l^*) > 0$. Further, the *MDD* cannot increase, so the new strategy makes K trades and has strictly greater Sterling ratio than S_{k-1}^K , which contradicts optimality of S_{k-1}^K . Thus, S_{k-1}^K cannot contain I_l^* as one of its intervals.

Thus $Q(k)$ holds for all $k \geq 1$, concluding the proof. ■

We are now ready to give the $O(n \log n)$ algorithm that establishes Theorem 1.3. First, we can compute the optimal strategy that makes only one trade in $O(n \log n)$ (Section 3.1), and compare this with the trivial strategy that makes no trades. It remains to compute the K -Sterling-optimal strategy and pick the best. We show how to do this in $O(n \log n)$ time.

First we obtain \mathcal{T}_μ^* in linear time. Suppose \mathcal{T}_μ^* makes $K_0 > K$ trades (as otherwise \mathcal{T}_μ^* is our solution). By Theorem 3.26, we only need to construct the pools P_0, P_1, \dots , maintaining the pool with the optimal Sterling ratio for its top K trades, as explained in the following algorithm.

- 1: Set $i = 0$; Sort (in decreasing order) the intervals from P_0 according to profit; Sort all the bridges with finite weight in increasing order. Let B_i be the minimum weight bridge in P_i ; Let strategy S_i consist of the top K intervals, and let $\text{Strl}_{opt} = \text{Strl}(S_i)$;
- 2: **while** P_i contains at least K intervals and at least one finite weight bridge **do**
- 3: **if** $B_i = B(I_l, I_r)$ is regular **then**
- 4: Regular merge to obtain P_{i+1} : remove I_l, I_r, B_i from the interval and bridge orderings, and add back $I = I_l \cup B_i \cup I_r$ into the interval ordering; compute $\mu(I)$ and $MDD(I)$;
- 5: Update neighboring bridge weights and re-insert them back into the bridge ordering.
- 6: **else if** $B_i = B(I_l, I_r)$ is left-regular **then**
- 7: Irregular merge to obtain P_{i+1} : Let B_- be the bridge left of B_i ; remove I_l, B_-, B_i from the interval and bridge orderings. Create the new bridge $B = B_- \cup I_l \cup B_i$, compute $W(B)$ and insert B into the bridge ordering (note that $W(B)$ may be ∞).
- 8: **end if**
- 9: $i \leftarrow i + 1$; update Strl_i ; if $\text{Strl}_{opt} < \text{Strl}_i$, then $\text{Strl}_{opt} \leftarrow \text{Strl}_i$.
- 10: **end while**

The correctness of the algorithm follows from Theorem 3.26. We now analyse the run time of an efficient implementation of the algorithm. P_0 contains at most n intervals and bridges. Each

execution of the while loop reduces loop number of bridges and intervals by 1 each, so the while loop is executed at most $n/2$ times. Merging two intervals is a constant time operation. The profit of a new interval is the profit of the merged intervals minus the weight of the merging bridge (also computable in constant time). The MDD of a new interval is the maximum of the MDD of the merged intervals and the weight of the merging bridge (also computable in constant time). The weight of a new bridge takes constant time to compute, and updating the weights of the neighbour bridges is a constant time operation provided that pointers are maintained to them. These pointers can be updated in constant time as well. Thus the run time within the while loop is dominated by inserting into the bridge or interval orderings. At most a constant number of such inserts into a *sorted list* need to be done, and each is an $O(\log n)$ operation [7]. To efficiently implement step 9, we maintain two sorted lists of the top K intervals in the algorithm, sorted according to return and MDD . These can be maintained in $O(\log K)$ operations. The first allows us to update the total return of the top K intervals in constant time, and the second allows us to update the MDD of the top K intervals (by taking the interval with largest MDD) in constant time. Thus the total running time of the while loop is $O(n \log n + n \log K) = O(n \log n)$. The preprocessing (step 1) is $O(n)$, and so does not contribute to the asymptotic run time.

4 Sharpe Optimal Trading Strategies

Another popular measure of the portfolio's risk-adjusted return is the Sharp Ratio. For trading strategy \mathcal{T} , we consider two versions of the Sharpe ratio, Shrp_1 and Shrp_2 .

$$\text{Shrp}_1(\mathcal{T}) = \frac{\mu(\mathcal{T})}{\sigma(\mathcal{T})}, \quad \text{Shrp}_2(\mathcal{T}) = \frac{\mu(\mathcal{T})}{\sigma^2(\mathcal{T})}. \quad (9)$$

Note that Shrp_2 is more conservative in that it penalizes large variances more heavily. We introduce a simplified Sharpe ratio (SSR) S that will be instrumental to finding the optimal strategies,

$$S = \frac{\mu}{s^2}.$$

It is easy to check that maximizing Shrp_1 is equivalent to maximizing $\frac{\mu^2}{s^2}$, and that Shrp_2 is given by $\frac{\bar{r}}{\frac{1}{n}s^2 - \bar{r}^2}$, where \bar{r} is the mean return. We will relate the maximization of Shrp_1 and Shrp_2 to the maximization of S .

Let \mathcal{T} be a trading strategy that makes K trades, with trading intervals I_1, \dots, I_K . Each trade contributes a transactions cost of $-f_{sp}$ to the return sequence. In general, a trade contributes $\hat{f}_S^2 + \hat{f}_B^2$ to s^2 . However, we will assume that $f_{sp} \ll 1$ and so we will ignore the contribution of the transactions cost to s^2 . Alternatively, we can justify this by assuming that the transactions cost is

spread finely over many time intervals. The sum over these small time intervals is finite, equal to $-f_{sp}$, however, the sum of squares over these small time intervals can be made arbitrarily small. Define the total return and sum of squared returns for each trading interval,

$$\mu_i = \mu(I_i) = \sum_{j \in I_i} r[j], \quad s_i^2 = s^2(I_i) = \sum_{j \in I_i} r[j]^2.$$

We define A_i as the contribution of trade i to the mean return, and B_i as the contribution of trade i to the mean squared return (ignoring the effect of the transactions cost), i.e., $A_i = \frac{1}{n}(\mu_i - f_{sp})$ and $B_i = \frac{1}{n}s_i^2$. We define $A(\mathcal{T}) = \sum_{k=1}^K A_i$ (note that $\bar{r} = A(\mathcal{T})$) and $B(\mathcal{T}) = \sum_{k=1}^K B_i$ (note that $\frac{1}{n}s^2 = B(\mathcal{T})$).

4.1 Maximizing the Simplified Sharpe Ratio S

We will need the following technical lemma, which can be proved by an easy induction.

Lemma 4.1 *Let $\mathcal{F} = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_k}{b_k}\}$ be any set of fractions satisfying $b_i > 0$ and $\frac{c}{d} \leq \frac{a_i}{b_i} \leq \frac{a}{b}$, for all i , where $b, d > 0$. Then, $\frac{c}{d} \leq \frac{a_1 + a_2 + \dots + a_k}{b_1 + b_2 + \dots + b_k} \leq \frac{a}{b}$. The upper (resp. lower) bound is strict if at least one of the fractions in \mathcal{F} is strictly upper (resp. lower) bounded by $\frac{a}{b}$ (resp. $\frac{c}{d}$).*

Let \mathcal{T}^* be an SSR-optimal strategy making $K > 1$ trades with trading intervals I_1, \dots, I_K .

$$S(\mathcal{T}^*) = \frac{\sum_{i=1}^K A_i}{\sum_{i=1}^K B_i} = \frac{A(\mathcal{T}^*)}{B(\mathcal{T}^*)},$$

Lemma 4.2 $\frac{A_i}{B_i}$ is a constant for every interval i , i.e., every trade is equivalent.

Proof: Suppose that $\min_i \frac{A_i}{B_i} < \frac{A_j}{B_j}$ for some j (strict inequality), and without loss of generality, assume that the minimum is attained for interval I_1 . By Lemma 4.1, if we remove I_1 , we get that

$$S(I_1 \cup \dots \cup I_K) = \frac{\sum_{i=1}^K A_i}{\sum_{i=1}^K B_i} < \frac{\sum_{i=2}^K A_i}{\sum_{i=2}^K B_i} = S(I_2 \cup \dots \cup I_K),$$

which contradicts the optimality of \mathcal{T}^* implying that $\min_i \frac{A_i}{B_i} = \frac{A_j}{B_j}$ for all j . ■

Corollary 4.3 *An SSR-optimal trading strategy making one trade exists.*

Proposition 4.4 *An SSR-optimal strategy making one trade can be found in $O(n \log n)$ time.*

Proof: By Corollary 4.3, we are guaranteed the existence of such a strategy. It suffices to find the single interval I maximizing $\sum_{i \in I} r[i] / \sum_{i \in I} r[i]^2$. Consider all intervals starting at position i and define $c_k = \sum_{j=i}^k r[j]$ and $d_k = \sum_{j=i}^k r[j]^2$. We wish to find k to maximize c_k/d_k . If we have

done this for position i , we now consider position $i - 1$. We show that the algorithm in Section 3.1.1 can be used. Trade intervals starting at $i - 1$ correspond to shifting all the c_k by $r[i - 1]$, and all the d_k by $r[i - 1]^2$. Both these operations simply correspond to shifting the origin point \mathbf{p} to $\mathbf{p}' = \mathbf{p} - (r[i - 1], r[i - 1]^2)$. We then add a new leftmost point at \mathbf{p} . Since each update takes $O(\log n)$, and the optimal interval for the new points can be found in $O(\log n)$, the entire algorithm runs in $O(n \log n)$. \blacksquare

4.2 Maximizing Shrp_2

Ignoring the f_{sp}^2 term in the denominator changes the denominator slightly, so we introduce the slightly different quantity $\overline{\text{Shrp}}_2$. Specifically,

$$\text{Shrp}_2(\mathcal{T}) = \frac{A(\mathcal{T})}{\frac{d}{n} f_{sp}^2 + B(\mathcal{T}) - A^2(\mathcal{T})}, \quad \overline{\text{Shrp}}_2(\mathcal{T}) = \frac{A(\mathcal{T})}{B(\mathcal{T}) - A^2(\mathcal{T})},$$

where d is the number of trades in \mathcal{T} . By the Cauchy-Schwarz inequality, for any trading strategy, $\sum r[i]^2 \geq \frac{1}{n} (\sum_i r[i])^2$. Since we are only interested in strategies for which $A(\mathcal{T}) \geq 0$, we have

Lemma 4.5 *For strategy \mathcal{T} , if $A(\mathcal{T}) > 0$ then $B(\mathcal{T}) - A^2(\mathcal{T}) > 0$*

We will show that maximizing $\overline{\text{Shrp}}_2$ is closely related to a constrained optimization of the SSR, and that maximizing $\overline{\text{Shrp}}_2$ is not too far from maximizing Shrp_2 .

Let \mathcal{T}_μ^* be return optimal, with return $\mu(\mathcal{T}_\mu^*) = \mu^*$. For any $0 \leq \alpha \leq \mu^*$, we define the constrained SSR-optimal strategy \mathcal{T}_α as the strategy with maximum SSR among all strategies with return at least α , i.e., $A(\mathcal{T}_\alpha) \geq \alpha$ and for all strategies \mathcal{T} with $A(\mathcal{T}) \geq \alpha$, $S(\mathcal{T}_\alpha) \geq S(\mathcal{T})$. Note that while an SSR-optimal strategy can be chosen with one trading interval, a constrained SSR-optimal strategy may require more than one trading interval. We show that for some appropriate threshold α , the constrained SSR-optimal strategy is a $\overline{\text{Shrp}}_2$ -optimal strategy.

Proposition 4.6 $\exists 0 \leq \alpha \leq \mu^*$ such that the constrained SSR-optimal strategy \mathcal{T}_α is $\overline{\text{Shrp}}_2$ -optimal.

Proof: Let \mathcal{T} be any $\overline{\text{Shrp}}_2$ -optimal strategy, and let $\alpha^* = A(\mathcal{T})$. Let \mathcal{T}_{α^*} be any constrained SSR-optimal strategy. Then $A(\mathcal{T}_{\alpha^*}) \geq A(\mathcal{T})$ and since $S(\mathcal{T}_{\alpha^*}) \geq S(\mathcal{T})$, we have that

$$0 \leq A(\mathcal{T}_{\alpha^*})B(\mathcal{T}) - A(\mathcal{T})B(\mathcal{T}_{\alpha^*}).$$

Suppose that $\overline{\text{Shrp}}_2(\mathcal{T}_{\alpha^*}) < \overline{\text{Shrp}}_2(\mathcal{T})$, then

$$0 \leq A(\mathcal{T}_{\alpha^*})B(\mathcal{T}) - A(\mathcal{T})B(\mathcal{T}_{\alpha^*}) < A(\mathcal{T}_{\alpha^*})A(\mathcal{T}) \cdot (A(\mathcal{T}) - A(\mathcal{T}_{\alpha^*})).$$

Both $A(\mathcal{T}_{\alpha^*})$ and $A(\mathcal{T})$ are > 0 , otherwise both strategies are inferior to \mathcal{T}_{μ^*} ; thus $A(\mathcal{T}) > A(\mathcal{T}_{\alpha^*})$, which is a contradiction. Therefore $\overline{\text{Shrp}}_2(\mathcal{T}_{\alpha^*}) \geq \overline{\text{Shrp}}_2(\mathcal{T})$ and so \mathcal{T}_{α^*} is $\overline{\text{Shrp}}_2$ -optimal. \blacksquare

We will need the following property of any SSR-optimal interval.

Proposition 4.7 *Let J be a subinterval of an SSR-optimal interval I . Then, $\mu(J) \geq -f_{sp}$. Further, if J is a prefix or suffix of I , then $\mu(J) > 0$.*

Proof: If J is a prefix or suffix of I and $\mu(J) \leq 0$, then deleting J from I gives at least as much return, with smaller sum of squared returns, contradicting the SSR-optimality of I . Suppose that $I = L \cup J \cup R$ where L and R are nonempty subintervals of I . If $\frac{\mu(J) + \mu(R)}{s^2(J) + s^2(R)} < \frac{-f_{sp} + \mu(L)}{s^2(L)}$, then by Lemma 4.1,

$$S(I) = \frac{-f_{sp} + \mu(L) + \mu(J) + \mu(R)}{s^2(L) + s^2(J) + s^2(R)} < \frac{-f_{sp} + \mu(L)}{s^2(L)} = S(L) \quad (*)$$

This contradicts the optimality of I , so we have

$$\frac{\mu(J) + \mu(R)}{s^2(J) + s^2(R)} \geq \frac{-f_{sp} + \mu(L)}{s^2(L)}.$$

Now, suppose that $\mu(J) < -f_{sp}$. Using (*) and Lemma 4.1, we find that

$$S(I) \leq \frac{\mu(J) + \mu(R)}{s^2(J) + s^2(R)} < \frac{-f_{sp} + \mu(R)}{s^2(J) + s^2(R)} < \frac{-f_{sp} + \mu(R)}{s^2(R)} = S(R),$$

because $s^2(J) > 0$. This contradicts the SSR-optimality of I , so $\mu(J) \geq -f_{sp}$. \blacksquare

We show that adding an SSR-optimal interval to any trading strategy can only improve the strategy.

Proposition 4.8 *Let I_0 be any SSR-optimal interval, and let \mathcal{T} be any trading strategy. Let $\mathcal{T}' = I_0 \cup \mathcal{T}$. Then $A(\mathcal{T}') \geq A(\mathcal{T})$ and $S(\mathcal{T}') \geq S(\mathcal{T})$*

Proof: If I_0 is contained in some interval of \mathcal{T} , then there is nothing to prove, so suppose that $\mathcal{T} \cup I_0 \neq \mathcal{T}$, and that \mathcal{T} contains $d \geq 1$ trades I_1, \dots, I_d . Note that $S(\mathcal{T}) = A(\mathcal{T})/B(\mathcal{T})$. If I_0 and \mathcal{T} do not intersect, then $\mathcal{T}' = I_0 \cup \mathcal{T} = \{I_0, I_1, \dots, I_d\}$. $A(\mathcal{T}') = A(\mathcal{T}) + A(I_0) \geq A(\mathcal{T})$, because $A(I_0) \geq 0$. Since I_0 is SSR-optimal, $S(I_0) = \frac{A(I_0)}{B(I_0)} \geq \frac{A(\mathcal{T})}{B(\mathcal{T})} = S(\mathcal{T})$, so by lemma 4.1,

$$S(\mathcal{T}') = \frac{A(\mathcal{T}')}{B(\mathcal{T}')} = \frac{A(\mathcal{T}) + A(I_0)}{B(\mathcal{T}) + B(I_0)} \geq \frac{A(\mathcal{T})}{B(\mathcal{T})} = S(\mathcal{T}).$$

Suppose that $\mathcal{T} \cap I_0 \neq \emptyset$. We can decompose \mathcal{T} into four parts (each part could be empty): $\mathcal{T} = S_1 \cup S_2 \cup I_l \cup I_r$, where S_1 contains intervals that do not intersect with I_0 , S_2 contains intervals that are contained in I_0 , I_l is not contained in I_0 but overlaps I_0 on the left, and I_r is not contained

in I_0 but overlaps I_0 on the right. $\mathcal{T}' = I_0 \cup \mathcal{T} = S_1 \cup I_l \cup I_0 \cup I_r$, i.e., adding I_0 combines all the trades in $\{S_2, I_l, I_r\}$ into one trade. Since the internal regions of I_0 have return at least $-f_{sp}$ and any prefix and suffix of I_0 has positive return (Proposition 4.7), we see that merging any two consecutive trades overlapping I_0 decreases the number of trades by one, hence increases the return by f_{sp} and the added interval loses at most f_{sp} , hence this merge can only increase $A(\mathcal{T})$. If either I_l or I_r are empty, then we are additionally adding a prefix or suffix of I_0 without changing the number of trades, which also increases $A(\mathcal{T})$, thus we see that $A(\mathcal{T}') \geq A(\mathcal{T})$.

Let's introduce the following definitions,

$$\begin{aligned} A_1 &= A(S_1) + \frac{1}{n}(\mu(I_l \cap \overline{I_0}) + \mu(I_r \cap \overline{I_0})) \\ A_2 &= A(S_2) + \frac{1}{n}(\mu(I_l \cap I_0) - f_{sp} + \mu(I_r \cap I_0) - f_{sp}) \\ B_1 &= B(S_1) + \frac{1}{n}(s^2(I_l \cap \overline{I_0}) + s^2(I_r \cap \overline{I_0})) \\ B_2 &= B(S_2) + \frac{1}{n}(s^2(I_l \cap I_0) + s^2(I_r \cap I_0)), \end{aligned}$$

where $\overline{I_0}$ is the complement of I_0 . Letting $A_0 = A(I_0)$ and $B_0 = B(I_0)$, we then have

$$\mathbb{S}(\mathcal{T}) = \frac{A_1 + A_2}{B_1 + B_2}, \text{ and } \mathbb{S}(\mathcal{T}') = \frac{A_1 + A_0}{B_1 + B_0}.$$

Note that $\mathbb{S}(S_2 \cup (I_l \cap \overline{I_0}) \cup (I_r \cap \overline{I_0})) = \frac{A_2}{B_2}$, so by the optimality of I_0 , $\frac{A_2}{B_2} \leq \frac{A_0}{B_0}$. We show that

$$\frac{\frac{1}{n}\mu(I_l \cap \overline{I_0})}{\frac{1}{n}s^2(I_l \cap \overline{I_0})} \leq \frac{A_0}{B_0}, \text{ and } \frac{\frac{1}{n}\mu(I_r \cap \overline{I_0})}{\frac{1}{n}s^2(I_r \cap \overline{I_0})} \leq \frac{A_0}{B_0}. \quad (**)$$

If not, then suppose (for example) that $\frac{\frac{1}{n}\mu(I_l \cap \overline{I_0})}{\frac{1}{n}s^2(I_l \cap \overline{I_0})} > \frac{A_0}{B_0}$. Then,

$$\mathbb{S}(I_0 \cup I_l) = \frac{A_0 + \frac{1}{n}\mu(I_l \cap \overline{I_0})}{B_0 + \frac{1}{n}s^2(I_l \cap \overline{I_0})} > \frac{A_0}{B_0} = \mathbb{S}(I_0)$$

contradicting the SSR-optimality of I_0 . Again, by the optimality of I_0 , $\mathbb{S}(S_1) = \frac{A(S_1)}{B(S_2)} \leq \frac{A_0}{B_0} = \mathbb{S}(I_0)$. it also has to be sharper than strategy S_1 . Thus, using $(**)$ and Lemma 4.1 we have that $\frac{A_1}{B_1} \leq \frac{A_0}{B_0}$. Because A_2 is obtained from the returns of a collection of subintervals of I_0 , it follows from Proposition 4.7 that $A_2 \leq A_0$. Now suppose that $\mathbb{S}(\mathcal{T}) > \mathbb{S}(\mathcal{T}')$, i.e.,

$$(A_1 + A_2)(B_1 + B_0) - (A_1 + A_0)(B_1 + B_2) > 0.$$

Since $A_2 \leq A_0$, it follows that $B_2 \leq B_0$. Rearranging terms in the equation above, we have that

$$\begin{aligned} \frac{A_2 B_1 - A_1 B_2}{B_0(B_1 + B_2)} &> \frac{A_0}{B_0} - \frac{A_1 + A_2}{B_1 + B_2}, \\ &\geq \frac{A_2}{B_2} - \frac{A_1 + A_2}{B_1 + B_2} = \frac{A_2 B_1 - A_1 B_2}{B_2(B_1 + B_2)}. \end{aligned}$$

Since $S(I_0) \geq S(T)$, the first inequality shows that $A_2 B_1 - A_1 B_2 > 0$. The second inequality then implies that $B_2 > B_0$, a contradiction. \blacksquare

We can now give the intuition behind our algorithm. The starting point is Proposition 4.6, which says that it suffices to look for constrained SSR-optimal strategies. So the natural first choice is an unconstrained SSR-optimal interval \mathcal{T}_0 . Either this will be $\overline{\text{Shrp}}_2$ optimal or not. If not, it is because it has too small a return. So our next step is to add to this interval an new interval (possibly adjacent) with the property that the interval increases the return with *smallest* possible decrease in SSR, resulting in strategy \mathcal{T}_1 . We repeat this process, constructing a sequence of trading strategies $\mathcal{T}_0, \mathcal{T}_1, \dots$ with the property that $A(\mathcal{T}_i) > A(\mathcal{T}_{i-1})$, and among all other strategies \mathcal{T} such that $A(\mathcal{T}) > A(\mathcal{T}_{i-1})$, $S(\mathcal{T}_i) \geq S(\mathcal{T})$. We then pick the strategy \mathcal{T}_{i^*} with maximum $\overline{\text{Shrp}}_2$ ratio among these strategies, which will be globally sharpe optimal.

Suppose that we have a current strategy, \mathcal{T}_i . We need to determine the next piece to add to this so that we increase the return, with smallest possible decrease in SSR. Let \mathcal{T}_i be composed of the intervals I_0, I_1, \dots, I_d . We replace each of these intervals by a special symbol, \$, to signify that these regions are already included in the strategy. We thus obtain a *generalized returns sequence*, one in which some intervals are replaced by the \$ symbol. A *generalized trading strategy* on the generalized return sequence must be composed of trades that do not contain the \$ symbol. However trades may be adjacent to the \$ symbol. A trade interval I in a generalized trading strategy can be *isolated* (not adjacent to any \$ symbol), *extending* (adjacent to one \$ symbol), or *bridging* (adjacent to two \$ symbols). In order to correctly account for the transactions cost, we need to change how we compute $A(I)$, so we introduce the new function $\bar{A}(I)$:

$$\bar{A}(I) = \begin{cases} A(I) & \text{I is isolated} \\ A(I) + \frac{f_{sp}}{n} & \text{I is extending} \\ A(I) + \frac{2f_{sp}}{n} & \text{I is bridging} \end{cases}$$

The *generalized simplified Sharp ratio (GSSR)* for generalized strategy $\mathcal{T} = \{I_1, \dots, I_d\}$ is

$$\bar{S}(\mathcal{T}) = \frac{\sum_{i=1 \dots d} \bar{A}(I_i)}{\sum_{i=1 \dots d} B(I_i)}$$

Similar to the notion of a maximal return optimal strategy, we introduce the notion of a maximal SSR-optimal (or GSSR-optimal) interval as one which cannot be extended in either direction without decreasing the SSR (or GSSR).

We now define generalized return sequences $\{R_0, R_1, \dots\}$ as follows. R_0 is just the original returns sequence. Let I_i be a maximal GSSR-optimal interval for R_i . We obtain the generalized sequence R_{i+1} by replacing $I_i \subset R_i$ with the symbol $\$$. We define any set of generalized sequences obtained in this way as *monotone*. We also refer to a member of a monotone set as monotone. Let R_0, R_1, \dots, R_k be a monotone sequence of generalized returns sequences, and let I_0, I_1, \dots, I_k be the maximal GSSR-optimal intervals corresponding to each sequence. By construction, I_i is a maximal GSSR-optimal interval for R_i . We have defined \bar{A} so that the SSR of the union of these intervals in R_0 is given by

$$S_{R_0}(I_0 \cup I_1 \cup \dots \cup I_k) = \frac{\sum_{i=1}^d \bar{A}_{R_i}(I_i)}{\sum_{i=1}^d B(I_i)},$$

where the subscript R_i indicates on which generalized return sequence the quantity is computed.

Lemma 4.9 $\bar{S}_{R_i}(I_i) \geq \bar{S}_{R_{i+1}}(I_{i+1})$

Proof: Suppose that $\bar{S}_{R_i}(I_i) < \bar{S}_{R_{i+1}}(I_{i+1})$, and let $\$$ be the symbol that replaced I_i in R_i to obtain R_{i+1} . If I_{i+1} is not adjacent with $\$$, then I_i is not GSSR-optimal in R_i , a contradiction. If I_{i+1} is adjacent with $\$$, then $I_i \cup I_{i+1}$ has higher GSSR (by Lemma 4.1), so once again I_i is not GSSR-optimal in R_i . ■

Now an easy induction, using Lemmas 4.1 and 4.9 gives,

Corollary 4.10 $S_{R_0}(I_0 \cup I_1 \cup \dots \cup I_k) \geq \bar{S}_{R_k}(I_k)$ for any k .

Analogous to Propositions 4.4, 4.7, 4.8, we have the following three propositions. Their proofs are almost identical, so we omit them.

Proposition 4.11 *A GSSR-optimal strategy making one trade exists, and all maximal GSSR-optimal trades can be found in $O(N \log N)$ time.*

Proposition 4.12 *Let J be a subinterval of any GSSR-optimal interval I . Then $\mu(J) \geq -f_{sp}$. If J is a prefix or suffix of I that is not adjacent with the symbol "\$", then $\mu(J) > 0$.*

Proposition 4.13 *Let I_0 be any GSSR-optimal interval, and let \mathcal{T} be any generalized trading strategy. Let $\mathcal{T}' = I_0 \cup \mathcal{T}$. Then, $\bar{A}(\mathcal{T}') \geq \bar{A}(\mathcal{T})$ and $\bar{S}(\mathcal{T}') \geq \bar{S}(\mathcal{T})$.*

We now give the main result that will lead to the final algorithm to obtain the $\overline{\text{Shrp}}_2$ -optimal strategy. Its essential content is that given a monotone set of generalized returns sequences, R_0, R_1, \dots , with corresponding GSSR-optimal intervals I_0, I_1, \dots , for some k , $\mathcal{T} = I_0 \cup I_1 \cup \dots \cup I_k$ is $\overline{\text{Shrp}}_2$ optimal. We will need some preliminary results.

Proposition 4.14 *For some k , $\mathcal{T}^* = I_0 \cup I_1 \cup \dots \cup I_k$ is $\overline{\text{Shrp}}_2$ -optimal., where I_i are the GSSR-optimal intervals corresponding to a monotone set of generalized returns sequences.*

Proof: First we show that there exists a $\overline{\text{Shrp}}_2$ -optimal strategy \mathcal{T}_0 that contains I_0 . Indeed, let \mathcal{T} be any $\overline{\text{Shrp}}_2$ -optimal strategy, and consider $\mathcal{T}_0 = \mathcal{I}_0 \cup \mathcal{T}$. By the Proposition 4.8, we have $S(\mathcal{T}_0) \geq S(\mathcal{T})$ and $A(\mathcal{T}_0) \geq A(\mathcal{T}) \geq 0$. Then,

$$\overline{\text{Shrp}}_2(\mathcal{T}_0) - \overline{\text{Shrp}}_2(\mathcal{T}) = \frac{A(\mathcal{T}_0)A(\mathcal{T})(A(\mathcal{T}_0) - A(\mathcal{T})) + B(\mathcal{T}_0)B(\mathcal{T})(S(\mathcal{T}_0) - S(\mathcal{T}))}{(B(\mathcal{T}_0) - A^2(\mathcal{T}_0))(B(\mathcal{T}) - A^2(\mathcal{T}))} \geq 0,$$

thus, \mathcal{T}_0 is $\overline{\text{Shrp}}_2$ -optimal.

Let \mathcal{T}_k be a $\overline{\text{Shrp}}_2$ -optimal strategy that contains $I_0 \cup \dots \cup I_k$. We know that \mathcal{T}_0 exists. If $\mathcal{T}_k = I_0 \cup \dots \cup I_k$, then we are done. If, $\mathcal{T}_k = I_0 \cup \dots \cup I_k \cup \mathcal{T}'$, with $\mathcal{T}' \neq \emptyset$, then we show that there must exist a $\overline{\text{Shrp}}_2$ -optimal strategy \mathcal{T}_{k+1} which contains $I_0 \cup \dots \cup I_{k+1}$, i.e., there is some other $\mathcal{T}'' \supseteq I_{k+1}$ such that $\mathcal{T}_{k+1} = I_0 \cup \dots \cup I_k \cup \mathcal{T}''$ is $\overline{\text{Shrp}}_2$ -optimal. The proposition then follows by an easy induction.

Let $\mathcal{T}'' = \mathcal{T}' \cup I_{k+1}$. Then, $\bar{A}_{R_{k+1}}(\mathcal{T}'') \geq \bar{A}_{R_{k+1}}(\mathcal{T}')$ and $\bar{S}_{R_{k+1}}(\mathcal{T}'') \geq \bar{S}_{R_{k+1}}(\mathcal{T}')$ (Proposition 4.13). By Corollary 4.10 and the GSSR-optimality of I_{k+1} , we have that

$$S(I_0 \cup \dots \cup I_k) \geq \bar{S}_{R_{k+1}}(I_{k+1}) \geq \bar{S}_{R_{k+1}}(\mathcal{T}'') \geq \bar{S}_{R_{k+1}}(\mathcal{T}')$$

From now on, we will drop the R_{k+1} subscript. Let $A = A(I_0 \cup \dots \cup I_k)$, $B = B(I_0 \cup \dots \cup I_k)$, $A' = \bar{A}(\mathcal{T}')$, $B' = B(\mathcal{T}')$, $A'' = \bar{A}(\mathcal{T}'')$ and $B'' = B(\mathcal{T}'')$. Let $\overline{\text{Shrp}}_2 = \overline{\text{Shrp}}_2(I_0 \cup \dots \cup I_k)$, $\overline{\text{Shrp}}_2' = \overline{\text{Shrp}}_2(I_0 \cup \dots \cup I_k \cup \mathcal{T}')$ and $\overline{\text{Shrp}}_2'' = \overline{\text{Shrp}}_2(I_0 \cup \dots \cup I_k \cup \mathcal{T}'')$. Thus,

$$\overline{\text{Shrp}}_2 = \frac{A}{B - A^2}, \quad \overline{\text{Shrp}}_2' = \frac{A + A'}{B + B' - (A + A')^2}, \quad \text{and} \quad \overline{\text{Shrp}}_2'' = \frac{A + A''}{B + B'' - (A + A'')^2}.$$

Let $A'' = \alpha'' A$ and $A' = \alpha' A$, where $\alpha'' \geq \alpha' > 0$. Then, by direct computation, one obtains

$$\overline{\text{Shrp}}_2' = \frac{A}{B + A^2 + \frac{\alpha'}{1+\alpha'}(B' - B - A^2)}, \quad \text{and} \quad \overline{\text{Shrp}}_2'' = \frac{A}{B + A^2 + \frac{\alpha''}{1+\alpha''}(B'' - B - A^2)},$$

Since $\overline{\text{Shrp}}_2' > \overline{\text{Shrp}}_2$, we conclude that $\frac{B'}{\alpha'} - B - A^2 < 0$. Since $\bar{S}(\mathcal{T}'') \geq \bar{S}(\mathcal{T}')$, we have that

$\frac{B'}{\alpha'} \geq \frac{B''}{\alpha''}$, and since $\alpha'' \geq \alpha' > 0$, $\frac{\alpha''}{1+\alpha''} \geq \frac{\alpha'}{1+\alpha'} > 0$, therefore

$$\frac{\alpha''}{1+\alpha''} \left(\frac{B''}{\alpha''} - B - A^2 \right) \leq \frac{\alpha'}{1+\alpha'} \left(\frac{B'}{\alpha'} - B - A^2 \right) < 0,$$

and so $\overline{\text{Shrp}}_2'' \geq \overline{\text{Shrp}}_2'$, concluding the proof. \blacksquare

By Proposition 4.14, a $\overline{\text{Shrp}}_2$ -optimal trading strategy can be obtained by constructing the strategies \mathcal{T}_k , and then picking the one with the maximum value for $\overline{\text{Shrp}}_2$. The next proposition shows that this can be done in $O(N^2 \log N)$ time.

Proposition 4.15 *A $\overline{\text{Shrp}}_2$ -optimal trading strategy can be found in time $O(n^2 \log n)$.*

Proof: I_i can be obtained in $O(n \log n)$ time (Proposition 4.11). Since there are at most n such intervals (since each must be non-empty), obtaining all the intervals is in $O(n^2 \log n)$.

Given the intervals, a single scan can be used to obtain the k for which \mathcal{T}_k is $\overline{\text{Shrp}}_2$ -optimal. \blacksquare

One can improve the runtime to $O(n^2)$ if $O(n^2)$ memory is available, however, we do not discuss the details.

4.2.1 Approximation Ratio

We have given an algorithm that obtains a $\overline{\text{Shrp}}_2$ -optimal strategy. A modification to the algorithm constructs the hierarchy \mathcal{T}_i and pick the one with the highest one with value of Shrp_2 . Suppose we have a $\overline{\text{Shrp}}_2$ -optimal strategy \mathcal{T} and let \mathcal{T}^* be a Shrp_2 -optimal strategy. Then by Proposition 4.6, it must be that $A(\mathcal{T}^*) > A(\mathcal{T})$ and that $S(\mathcal{T}^*) \leq S(\mathcal{T})$. Since $\overline{\text{Shrp}}_2(\mathcal{T}) \geq \overline{\text{Shrp}}_2(\mathcal{T}^*)$, we have that $A^*(B - A^2) - A(B^* - A^{*2}) \leq 0$, where $A = A(\mathcal{T})$, $A^* = A(\mathcal{T}^*)$, $B = B(\mathcal{T})$, $B^* = B(\mathcal{T}^*)$. We can evaluate $\text{Shrp}_2(\mathcal{T}^*) - \text{Shrp}_2(\mathcal{T})$ to obtain

$$0 \leq \text{Shrp}_2(\mathcal{T}^*) - \text{Shrp}_2(\mathcal{T}) \leq \text{Shrp}_2^* \cdot \frac{\frac{d}{n} f_{sp}^2}{\frac{d}{n} f_{sp}^2 + B - A^2}$$

When $B - A^2 = O(1)$, which is usually the case, we see that this is a very accurate approximation (since $f_{sp} \ll 1$).

4.3 Maximizing Shrp_1

Once again, we introduce the slightly different quantity $\overline{\text{Shrp}}_1$,

$$\text{Shrp}_1(\mathcal{T}) = \frac{A(\mathcal{T})}{\sqrt{\frac{d}{n} f_{sp}^2 + B(\mathcal{T}) - A^2(\mathcal{T})}}, \quad \overline{\text{Shrp}}_1(\mathcal{T}) = \frac{A(\mathcal{T})}{\sqrt{B(\mathcal{T}) - A^2(\mathcal{T})}}.$$

We will optimize $\overline{\text{Shrp}}_1(\mathcal{T})$. Since maximizing $\overline{\text{Shrp}}_1(\mathcal{T})$ is equivalent to minimizing $1/\overline{\text{Shrp}}_1^2(\mathcal{T})$ the problem reduces to maximizing

$$Q(\mathcal{T}) = \frac{A^2(\mathcal{T})}{B(\mathcal{T})}$$

The entire algorithm is analogous to that for maximizing $\overline{\text{Shrp}}_2$ in the previous section, we only need to prove the analogs of Propositions 4.6 and 4.14.

Proposition 4.16 $\exists 0 \leq \alpha \leq \mu^*$ such that the constrained SSR-optimal strategy \mathcal{T}_α is Q -optimal.

Proof: Let \mathcal{T} be $\overline{\text{Shrp}}_1$ -optimal, and let $\alpha^* = A(\mathcal{T})$. Let \mathcal{T}_{α^*} be a corresponding constrained SSR-optimal strategy. $A(\mathcal{T}_{\alpha^*}) \geq A(\mathcal{T})$ and $\frac{A(\mathcal{T}_{\alpha^*})}{B(\mathcal{T}_{\alpha^*})} \geq \frac{A(\mathcal{T})}{B(\mathcal{T})}$. Multiplying these two inequalities gives that $\frac{A^2(\mathcal{T}_{\alpha^*})}{B(\mathcal{T}_{\alpha^*})} \geq \frac{A^2(\mathcal{T})}{B(\mathcal{T})}$, i.e. \mathcal{T}_{α^*} is also Q -optimal. ■

Proposition 4.17 For some k , $\mathcal{T}^* = I_0 \cup I_1 \cup \dots \cup I_k$ is Q -optimal., where I_i are the GSSR-optimal intervals corresponding to a monotone set of generalized returns sequences.

Proof: The proof is very similar to the proof of Proposition 4.14. Let \mathcal{T} be Q -optimal, and let $\mathcal{T}_0 = I_0 \cup \mathcal{T}$. Then $A(\mathcal{T}_0) \geq A(\mathcal{T})$ and $S(\mathcal{T}_0) \geq S(\mathcal{T})$. Multiplying these two inequalities give that $Q(\mathcal{T}_0) \geq Q(\mathcal{T})$, or that \mathcal{T}_0 is also Q -optimal.

Let \mathcal{T}_k be a Q -optimal strategy that contains $I_0 \cup \dots \cup I_k$. Introduce $\mathcal{T}', \mathcal{T}'' = I_{k+1} \cup \mathcal{T}'$ as in the proof of Proposition 4.14. Let $Q = Q(I_0 \cup \dots \cup I_k)$, $Q' = Q(I_0 \cup \dots \cup I_k \cup \mathcal{T}')$ and $Q'' = Q(I_0 \cup \dots \cup I_k \cup \mathcal{T}'')$,

$$Q = \frac{A^2}{B}, \quad Q' = \frac{(A + A')^2}{B + B'}, \quad \text{and} \quad Q'' = \frac{(A + A'')^2}{B + B''}.$$

Following exactly the same logic as in the proof to Proposition 4.14, we only need to show that $Q'' \geq Q'$. Let $A'' = \alpha''A$ and $A' = \alpha'A$, where $\alpha'' \geq \alpha' > 0$. $\frac{A''}{B''} \geq \frac{A'}{B'}$ implies that $\frac{B''}{\alpha''} \leq \frac{B'}{\alpha'}$, and so $\frac{B''}{\alpha''(\alpha''+2)} \leq \frac{B'}{\alpha'(\alpha'+2)}$. By direct computation, one obtains

$$Q' = \frac{A^2}{B + \left(1 - \frac{1}{(1+\alpha')^2}\right) \left(\frac{B'}{\alpha'(\alpha'+2)} - B\right)}, \quad Q'' = \frac{A^2}{B + \left(1 - \frac{1}{(1+\alpha'')^2}\right) \left(\frac{B''}{\alpha''(\alpha''+2)} - B\right)}.$$

Since $Q' > Q$, it must be that $\frac{B'}{\alpha'(\alpha'+2)} - B < 0$. Since $\alpha'' \geq \alpha'$, $1 - \frac{1}{(1+\alpha'')^2} \geq 1 - \frac{1}{(1+\alpha')^2}$, so we have that

$$\left(1 - \frac{1}{(1+\alpha'')^2}\right) \left(\frac{B''}{\alpha''(\alpha''+2)} - B\right) \leq \left(1 - \frac{1}{(1+\alpha')^2}\right) \left(\frac{B'}{\alpha'(\alpha'+2)} - B\right) < 0,$$

which implies that $Q'' \geq Q'$. ■

4.3.1 Approximation Ratio

Once again, a modification to the algorithm constructs the hierarchy \mathcal{T}_i and picks the one with the highest one with value of Shrp_1 . Suppose we have a $\overline{\text{Shrp}}_1$ -optimal strategy \mathcal{T} and let \mathcal{T}^* be a Shrp_1 -optimal strategy. By direct computation, and using the fact that $\overline{\text{Shrp}}_1(\mathcal{T}) \geq \overline{\text{Shrp}}_1(\mathcal{T}^*) \implies A^{*2}B - A^2B^* \leq 0$, we get

$$0 \leq \text{Shrp}_1^2(\mathcal{T}^*) - \text{Shrp}_1^2(\mathcal{T}) \leq \text{Shrp}_1^2(\mathcal{T}^*) \frac{\frac{df_{sp}^2}{n}}{\frac{df_{sp}^2}{n} + B}$$

which gives an approximation ratio of $\sqrt{1 - O(f_{sp}^2)}$ when $B = O(1)$.

5 Discussion

Our main goal was to provide the theoretical basis for the computation of *a posteriori* optimal trading strategies, with respect to various criteria. The highlights of our contributions are that return and *MDD*-optimal strategies can be computed very efficiently, even with constraints on the number of trades. Sharpe optimal strategies prove to be much tougher to compute. However, for slightly modified notions of the Sharpe ratio, where one ignores the impact of bid-ask spread squared we can compute the optimal strategy efficiently. This is a reasonable approach since in most cases, the bid-ask spread is $\sim 10^{-4}$. We also show that this modified optimal strategy is not far from optimal with respect to the unmodified Sharpe ratio.

We have introduced a new technique for optimizing quotients over intervals of a sequence. This technique is based on relating the problem to convex set operations, and for our purposes has direct application to optimizing the *MDD*, the simplified Sharpe ratio (SSR), which is an integral component in optimizing the Sharpe ratio, and the Downside Deviation Ratio (DDR). This technique may be of more general use in optimizing other financially important criteria.

A natural open problem is whether Sharpe optimal strategies can be computed under constraints on the number of trades. We suspect that the monotone hierarchy we created with respect to the SSR has an important role to play, but the result has been elusive. Other directions of future research include the implementation of the algorithms in various markets to obtain, compare and learn from the optimal trading strategy.

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