A Maximum Likelihood Approach to Variance Estimation for a Brownian Motion Using the High, Low and Close

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Abstract

Volatility is time-varying by nature, hence, only the most recent data should be used to estimate volatility. It is therefore crucial to make utmost use of the scant information typically available in shorter time windows. We propose a volatility estimator using the high and the low information in addition to the close price, all of which are typically available to investors. The proposed estimator is based on a maximum likelihood approach. We present simulations that indicate that our estimator obtains consistently better performance than existing estimators on simulated data. In addition, our simulations on real price data demonstrate that our method produces more stable estimates. We also consider the effects of quantized prices and discretized time. We develop corrections to the high and low estimates that can be used in conjunction with the maximum likelihood estimator (or in fact any estimator that uses the observed high and low information).
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1 Introduction

The volatility of a financial instrument is a crucial parameter for a number of reasons. It enters as a parameter in pricing formulas for derivative instruments, and plays key roles in asset allocation, and risk management. As a result, considerable attention has been devoted to the accurate estimation of volatility. Because it is recognized that volatility is time varying, it is imperative to use only the most recent price observations to construct an estimate of the volatility. To obtain a good estimator, one should thus attempt to make the utmost use of the small number of observations available. In this paper, we propose an estimator that uses the high and low price information in addition to the closing price used by conventional estimators. In practice this would be of great interest because most historical data is quoted with both the high and low in addition to the close.

Volatility estimates using high and low prices have been considered to some extent in the literature. All previous studies have considered securities characterized by geometric Brownian motion or Brownian motion. Parkinson [4] shows that expectation of the high minus the low squared is proportional to σ^2, and thus constructs an estimate based on the high minus the low. Garman and Klass [3] define a quadratic function of the high, low and close, and derive the parameters of such a function that result in the estimate being unbiased, (their estimate is unbiased only in the case of zero drift). Rogers et al [6, 7, 5] propose another formula, and show that it is an unbiased estimate even for non-zero drift. The problem with these approaches is that they are not necessarily optimal estimates. In addition, they consider only one period (one day for example). By taking the average of the estimates over the days considered in the data set, unbiasedness of the estimates will prevail, but optimality will generally not be valid.

In this paper we propose a new estimate for both the volatility and the drift using the maximum likelihood approach. We derive an expression for the joint density of the maximum and the minimum of a Brownian motion, and construct a likelihood function that we maximize by a two-dimensional search. Finally, the fidelity of the high and low observations becomes questionable when the price can be quantized and the Brownian is only observed at discrete time intervals. We present an analysis of these phenomena and ways to correct for the systematic errors.

One of the advantages of the maximum likelihood approach lies in the the fact that it produces estimates that are asymptotically efficient. Further, if one assumes independence among the time periods, as is customarily the case, then multiple time periods can be incorporated by using the product likelihood function. In addition, within this probabilistic framework, it is straightforward to employ a fully Bayesian, decision theoretic approach, whereby, one enforces certain priors that one might have on the drift and the volatility (for example, in a risk averse world, the drift should be higher than the risk free rate, [1]).

This paper is organized as follows. First, we develop the maximum likelihood formulation of the problem followed by extensions to discrete time and price. We then present extensive simulations to compare our method to existing methods (the close estimator, Parkinson's estimator, Rogers' et al estimator, and the Garman–Klass estimator). Finally we demonstrate our method on real data. We compare our method for estimating the volatility using to the method of Parkinson [4] and to the estimate based on using the close price alone. Our simulations indicate that, the RMS prediction error of our estimator is about 2.6–2.7 times less than that of the estimator using the closing price alone. In practical terms, using the closing price alone would require about 35–40 days of data to obtain a comparable accuracy to our method on 5 days. Other methods that use the high-low information also obtain reductions in the RMS prediction error when compared to the close, but not by as much as our method. For comparison, Parkinson's method obtains a reduction by a factor of about 2.2. On real data, we demonstrate that our method obtains more stable (and hence more realistic) time varying volatility estimates.

2 Maximum Likelihood Approach

We consider an instrument that follows a standard Brownian motion:

$$dx_t = \mu dt + \sigma dw \tag{1}$$

where μ and σ denote respectively the drift and the volatility of the instrument. Usually, financial instruments are assumed to obey geometric Brownian motion, which can be converted to a standard Brownian motion using a transformation of variables. Let the instrument value at time 0 be $x_0 = x$. Consider for the time being the single period case, for example one day. We will derive a volatility estimator using the high and the low information. Later in the section, we will show how to extend this estimator in a straight forward manner to the multi-period case (several days in the example we consider).

Denote the high and the low for the period by h and l respectively, i.e.,

$$h = \sup_{0 \le t \le T} x(t) \tag{2}$$

$$l = \inf_{0 \le t \le T} x(t) \tag{3}$$

where T is the length of the period. The idea behind the proposed method is to evaluate the conditional density

$$p(h, l|x, \mu, \sigma, T) \tag{4}$$

and then obtain the μ and the σ that maximize this likelihood function. To obtain such a probability density, we revisit a classical result for the problem of first passage time of a Brownian motion with drift and with two absorbing boundaries [2]. In the first passage problem we have a Brownian motion and two boundaries h_1 and h_2 with $h_1 < x(0) < h_2$. The first passage time is the time till $x(t)$ first crosses either of the boundaries. The density function and the distribution function of the first passage time have been derived (see [2]), and they are in the form of a series. Let the distribution function be

$$F(T|h_1, h_2, x, \mu, \sigma) \tag{5}$$

which represents the probability that first passage occurs in the interval $[0, T]$. One can see that the distribution function corresponding to the required density $p(h, l|x, \mu, \sigma, T)$ with respect to the high and low random variables is equal to $1 - F$. Hence we can obtain the density p by differentiation as follows:

$$p(h, l|x, \mu, \sigma, T) = -\left.\frac{\partial^2}{\partial h_1 \partial h_2} F(T|h_1, h_2, x, \mu, \sigma)\right|_{h_1 = l, h_2 = h} \tag{6}$$

where the right hand side represents the second order partial derivative with respect to the two barrier levels evaluated at the low and high. Dominé [2] has computed a series expansion for exactly this first passage time distribution F, hence what remains is to compute the necessary derivatives. The formulas are tedious and their explicit form is given in the appendix, see equation (31) in section A, which is a series representation for the density $p(h, l|x, \mu, \sigma, T)$, and can be computed to any desired accuracy by taking sufficiently many terms. As a practical point, it is found that as the volatility decreases, more terms in the series should be computed to maintain the accuracy of the estimate.

Consider now the multi-day case. Assume one observes a set of prices, which consists of the open x_0 of the first day and the triples $\{h_i, l_i, c_i\}_{i=1}^{N}$ where i indexes the consecutive days for which one has data, and $h,\ l,\ c$ represent high, low, close respectivly. We assume that the close of any given day is the open of the next day, hence we can define the series of opens by $o_1 = x_0$ and $o_i = c_{i-1}$, for $i > 1$. (If there are gaps in the data between a close and the next open, then the contribution of this period to the likelihood function can be obtained from a Gaussian distribution, since there is no high and low information to exploit.) Because of the Markov property, the likelihood for the set of N days becomes the product of the likelihoods of each day. The log likelihood then becomes a sum and is given by the formula

$$L(\mu, \sigma) = \sum_{i=1}^{N} \log p(l_i, h_i|o_i, \mu, \sigma, T) \tag{7}$$

This represents the function that we wish to maximize with respect to μ and σ. The values of μ and σ that maximize (7) represent our estimates $\hat{\mu}$ and $\hat{\sigma}$.

3 Discrete Time and Price

While a maximum likelihood (or Bayesian) approach may be optimal, the volatility estimates that result in practical situations may contain errors that arise from two possible sources. The Brownian motion is not directly observed, the observed process being a random walk that samples the Brownian motion at regular (possibly large) finite intervals. This is true to first approximation with tick data and certainly the case with (for example) 5 minute data. In addition, the quoted price may be discretized. This occurs in most financial markets where quotes are usually to the nearest 1/16.

Suppose that the quoted (observed) high is h_o. Let the true high be h, then the difference $\delta = h - h_o$ will have some distribution. One does not expect the price quantization to significantly bias this difference, since intuitively one might argue that sometimes the rounding will result in a positive error and sometimes a negative. On the other hand, the time discretization produces an error that is systematic and can be significant. Since the actual high could occur between the times when the price is observed, h_o will always be below h. This can lead to significant underestimation of the volatility.

Here, we present a discussion of these discretization effects. We do not describe all the details since this is not the main theme of the paper, however we do present formulas that will be derived elsewhere.

3.1 Discretized Price

We make the approximation that in analysing the discretization effect, we can treat the maximum and minimum independently. The joint density of the maximum and the close of a Wiener process has been computed in [8].

$$f_T(h,c) = \frac{2(2h-c)}{(2\pi)^{1/2}(\sigma^2 T)^{3/2}} exp\left[-((2h-c)^2 - 2\mu c T + \mu^2 T)/2\sigma^2 T\right] \tag{8}$$

Using this density, one can compute the conditional density $f_T(h|c, h \in [h_o - \epsilon, h_o + \epsilon])$. This density could be used to augment the maximum likelihood density to account for the fact that the actual observed high is not the quoted high. Another, simpler approach, is to observe that only h and h^2 appear in our formulas, so one could use the expectations of these values instead. This requires computing the expectations for δ and δ^2. These can be computed from the conditional density, and, in the asymptotic limit $\epsilon \to 0$, are given by

$$E[\delta] = \frac{2\epsilon^2}{2h_o - c}\left[\frac{1}{3} - \frac{(2h_o - c)^2}{2\sigma^2 T}\right] + o(\epsilon^2) \qquad E[\delta^2] = \frac{\epsilon^2}{3} + o(\epsilon^2) \tag{9}$$

As already mentioned, we expect the effect of the rounding to be minor as can be observed from the fact that the first order correction is $O(\epsilon^2)$ which is small if ϵ is small.

The correction for the minimum can be obtained in a similar manner. The joint density for the magnitude of the minimum and the close can be obtained by changing $\mu \to -\mu$, $c \to -c$ in the density for the maximum. All the calculations are then analogous, and, if we write $l = l_o - \delta$, the expectations of δ and δ^2 are also given by (9).

3.2 Discretized Time

The effect of observing the Brownian motion at discrete time intervals is expected to have the systematic effect of under estimating the maximum and over estimating the minimum. We follow the approximation used in [6]. Once again, write $h = h_o + \delta$, where now δ is necessarily positive. A non-zero δ arises from the fact that the Brownian motion can fluctuate above h_o in the intervals where the Brownian is not observed. As a first approximation, we assume that the largest $\delta > 0$ occurs in one of the two intervals adjacent to the time at which h_o is observed, once again following [6]. Let τ be the size of the time interval. Then, it is necessary to obtain the distribution of the maximum of a Brownian process given that it is h_o at $t = 0$ and less than zero at $t = \pm\tau$. This conditional distribution for δ can be computed, and is given by

$$P[\delta \geq h] = F_\tau(h|\mu,\sigma) + F_\tau(h|-\mu,\sigma) - F_\tau(h|\mu,\sigma)F_\tau(h|-\mu,\sigma) \tag{10}$$

where

$$F_\tau(h|\mu,\sigma) = \frac{e^{2\mu h/\sigma^2}\phi\left(\frac{2h+\mu\tau}{(\sigma^2\tau)^{1/2}}\right)}{\phi\left(\frac{\mu\tau}{(\sigma^2\tau)^{1/2}}\right)} \qquad \text{and} \qquad \phi(x) = \frac{1}{(2\pi)^{1/2}}\int_x^\infty du\; e^{-u^2/2} \tag{11}$$

We present the derivation elsewhere as it is not essential to the intuition. Once again, the two approaches could be to either incorporate this function into the likelihood, or to replace h and h^2 in the formulas for the likelihood by their expectations. In order to do this, we need to compute the expected values of δ and δ^2. In the asymptotic limit when $\tau \to 0$, one finds that

$$E[\delta] = \left(\frac{\sigma^2\tau}{\pi}\right)^{1/2} + o(\tau^{1/2}) \qquad\qquad E\left[\delta^2\right] = \lambda_\delta(\sigma^2\tau) + o(\tau) \tag{12}$$

where $\lambda_\delta \approx 0.4511$. Comparison with [6] shows that this more accurate value of $E[\delta]$ is higher by a factor of about 1.25 and the more accurate value of $E[\delta^2]$ is higher by a factor of about 1.6. An exactly analogous calculation can be performed for the distribution of the minimum with identical results, and we do not repeat the exercise here.

4 Simulation Results

In our first simulation, we will assume that μ is known and does not need to be estimated. It is frequently the case that this assumption is made by equating the drift to the risk free rate (this can be done provided that $\mu >> \sigma^2$). In our simulations, we compare different estimators by looking at their estimates based on observed data over a window ranging from 5 days to 50 days. We obtain the RMS prediction error ($\sqrt{E[(\hat{\sigma}-\sigma)^2]}$) using 2000 realizations of each window size. For our simulations, we set $T = 1$, $\mu = 0.02$, $\sigma = 0.5$ and $x_0 = 0$. We assume that the drift μ is known and only the volatility σ needs to be estimated. Shown in first four columns of Table 1 is a comparison of four methods. The first method uses the close prices only and the estimate is given by

$$\hat{\sigma}_{close} = \sqrt{\frac{1}{NT}\sum_{i=1}^{N}(c_i - o_i - \mu T)^2} \tag{13}$$

Days used	RMS Prediction Error							
	μ known				μ unknown		$\mu = 0$	
	Close	Park	R–S	ML	Close	ML	G–K	ML
5	0.1597	0.0713	0.0642	0.0621	0.1752	0.0639	0.0591	0.0640
10	0.1090	0.0489	0.0448	0.0426	0.1152	0.0434	0.0399	0.0417
15	0.0900	0.0410	0.0375	0.0353	0.0930	0.0354	0.0337	0.0346
20	0.0781	0.0360	0.0317	0.0303	0.0808	0.0307	0.0292	0.0304
25	0.0702	0.0317	0.0289	0.0273	0.0709	0.0270	0.0260	0.0271
30	0.0645	0.0292	0.0270	0.0246	0.0654	0.0248	0.0233	0.0245
35	0.0605	0.0272	0.0245	0.0230	0.0615	0.0229	0.0224	0.0232
40	0.0556	0.0252	0.0227	0.0215	0.0559	0.0215	0.0205	0.0215
45	0.0526	0.0238	0.0215	0.0200	0.0534	0.0202	0.0192	0.0196
50	0.0499	0.0222	0.0204	0.0192	0.0505	0.0191	0.0186	0.0191

Table 1: Comparison of different volatility estimation methods.

The second method (Parkinson's estimator) uses only the high and low values [4], and the estimate is given by

$$\hat{\sigma}_{Park} = \sqrt{\frac{1}{4N \ln 2} \sum_{i=1}^{N} (h_i - l_i)^2} \tag{14}$$

The third method (the Rogers-Satchell estimator) uses the high low and close prices [6], and is given by

$$\sigma_{RS} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (h_i - o_i)(h_i - c_i) + (l_i - o_i)(l_i - c_i)} \tag{15}$$

The fourth method is our maximum likelihood based method,

$$\hat{\sigma}_{ML} = \underset{\nu}{\operatorname{argmax}}\, L(\mu, \nu) \tag{16}$$

We show the results of these simulations in the first four columns of Table 1. The results are also sumarised in Figure 1. From these results it is clear that our method produces a superior estimate.

One might note that the Parkinson and Rogers-Satchell estimates do not rely on knowledge of μ. Thus, one might argue that one should not assume that μ is known in comparing these estimators with our Maximum likelihood estimator. One can repeat the previous simulation without assuming that μ is known. In this case, both μ and σ need to be estimated – this is only relevant for the close estimator and our maximum likelihood estimator. The results of this simulation are shown in Table 1 in the "μ unknown" columns. These results are also summarised in Figure 2. Our method not only remains superior to all the other methods, but the fact that μ needs to be estimated as well has not significantly worsened the performance.

As one further comparison, we consider the special case of $\mu = 0$. For this particular condition,

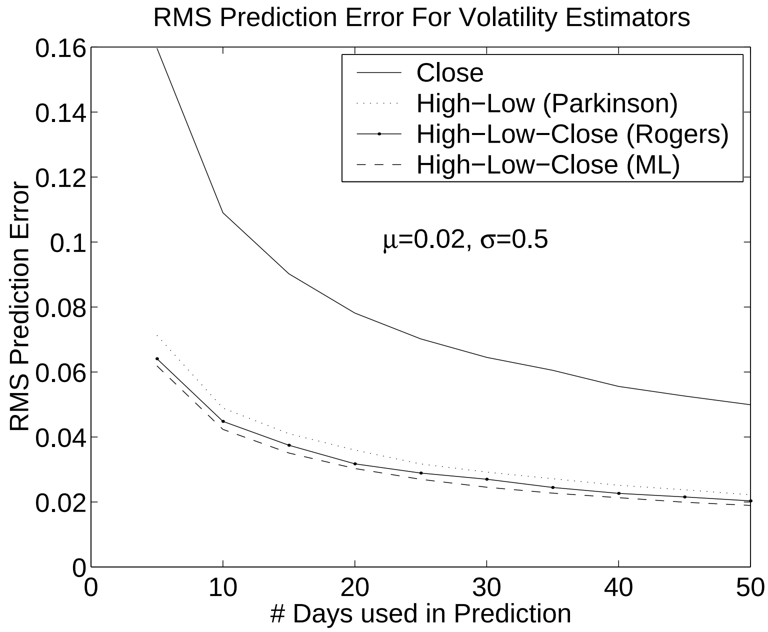

Figure 1: Comparison of volatility prediction methods using the RMS prediction error when the drift parameter is known.

Garman–Klass [3] have constructed the optimal (in the least squares sense) quadratic estimator as

$$\sigma_{GK} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} 0.511(\tilde{h}_i - \tilde{l}_i)^2 - 0.019(\tilde{c}_i(\tilde{h}_i + \tilde{l}_i) - 2\tilde{l}_i\tilde{h}_i) - 0.383\tilde{c}_i^2} \tag{17}$$

The tilde over the symbols indicates that one "normalizes" the quantities by subtracting the open prics. For example $\tilde{h}_i = h_i - o_i$. A comparison between this estimator and our maximum likelihood estimator for the case $\mu = 0$ is given in the last two columns of Table 1. In this special case of zero drift, our estimator does not beat the optimal quadratic estimator for small window sizes, but it approaches the optimal estimator as the window size increases. These results are summarised in Figure 3, along with the performance of the other estimators for this special case of $\mu = 0$. From Figure 3, it is seen that the Garman–Klass estimator is slightly better than the maximum likelihood estimator, but, however, one can also note that the maximum likelihood estimator is asymptotically approaching the Garman–Klass estimator as one might expect due to the asymptotic efficiency of maximum likelihood estimators. Also note that the Garman–Klass estimator only applies to the case of zero-drift.

We have also applied our method to real data. Since no ground truth is available in this case, we compare the methods based on stability measures for the volatility estimate. We estimate the volatility from a 10 day moving window using the close estimator, Parkinson's high–low estimator and our maximum likelihood high–low–close estimator. The volatility should not vary drastically from day to day. At most, it should be slowly varying, especially since the model assumes a constant volatility. We have experimented with several stocks and in all cases, the maximum likelihood

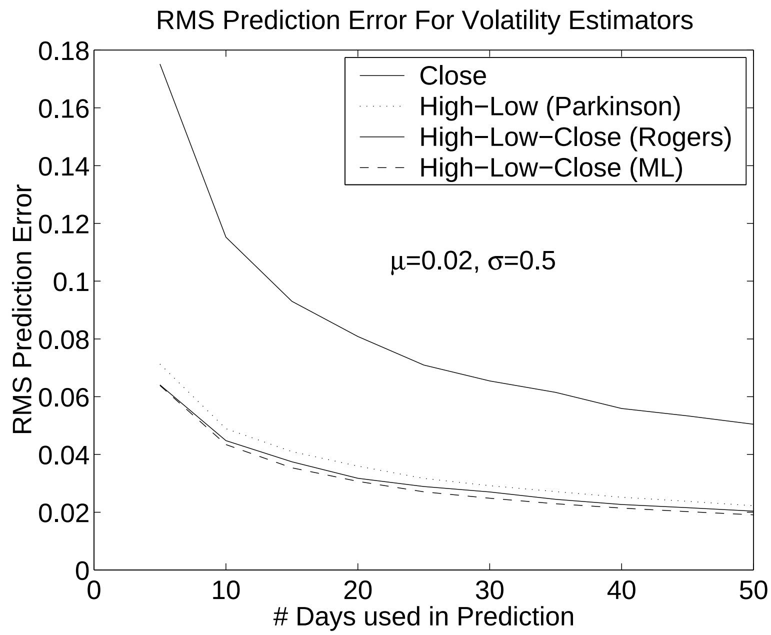

Figure 2: Comparison of volatility prediction methods using the RMS prediction error when the drift has to be estimated as well.

estimate is clearly the most stable. Representative volatility time series predictions based on a 10 day prediction window are shown for IBM in Figure 4. In general, it is desirable to use such a small prediction window as the volatility can change over time. The figures on the left show the volatility predictions. The figures on the right show the variability in the volatility predictions over a moving window of size 20 days. The figures on the left clearly show that the maximum likelihood estimate is qualitatively more stable, due in part to the fact that it is a more accurate estimate of the volatility. This qualitative difference is quantified in the figures on the right.

5 Conclusions

We have presented a formula for obtaining the joint distribution of the high and low given the open and the parameters of the Wiener process. Using this formula, one can construct a likelihood for the observed data given the parameters and hence obtain a maximum likelihood estimate. One could also employ a fully Bayesian framework to obtain a Bayes optimal estimator under some risk measure, if one has some prior information on the possible values of the volatility. We have shown that in simulations, our estimator obtains a significant improvement over the conventional close estimator as well as other estimators based on the high, low and close values. Further we have demonstrated that our method produces a much more stable estimator on real data, thus enabling one to make reliable volatility estimates using the few most recent data points. It is expected that a more accurate estimate of the time varying volatility will lead to more effecient pricing of volatility based derivative instruments such as options.

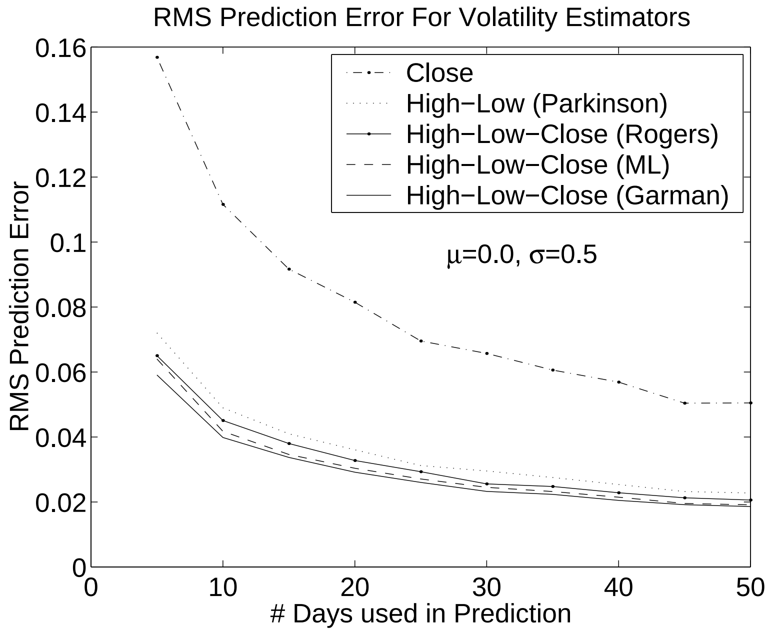

Figure 3: Comparison of volatility prediction methods using the RMS prediction error for the special case of zero drift.

A Joint Density of the High and Low

In this appendix, we compute a series expansion for the joint density of the high and the low given the open, the drift and volatility parameters. This is the expression that is needed for the computation of the likelihood as given in (7). We start with the distribution function (5). A series expansion is derived in [2] which we reproduce here for convenience.

$$F(T|h_1,h_2,x,\mu,\sigma) = \sum_{k=1}^{\infty} 2\sigma^4 k\pi \tilde{C}(k,x,\mu,\sigma,h_1,h_2) f(k,\mu,\sigma,T,(h_1-h_2)^2) \tag{18}$$

where the functions $\tilde{C}$ and f are given by

$$\begin{aligned}\tilde{C}(k,x,\mu,\sigma,h_1,h_2) &= \left[exp\Big(\frac{\mu}{\sigma^2}(h_2-x)\Big)(-1)^{k+1} + exp\Big(\frac{\mu}{\sigma^2}(h_1-x)\Big)\right] \\ &\quad \times \sin\left(k\pi\frac{x-h_1}{h_2-h_1}\right)\end{aligned} \tag{19}$$

and

$$f(k,\mu,\sigma,T,u) = \frac{exp\left(-\frac{g(k,\mu,\sigma,u)T}{2\sigma^2 u}\right)}{g(k,\mu,\sigma,u)} \tag{20}$$

where

$$g(k,\mu,\sigma,u) = \mu^2 u + \sigma^4 k^2 \pi^2 \tag{21}$$

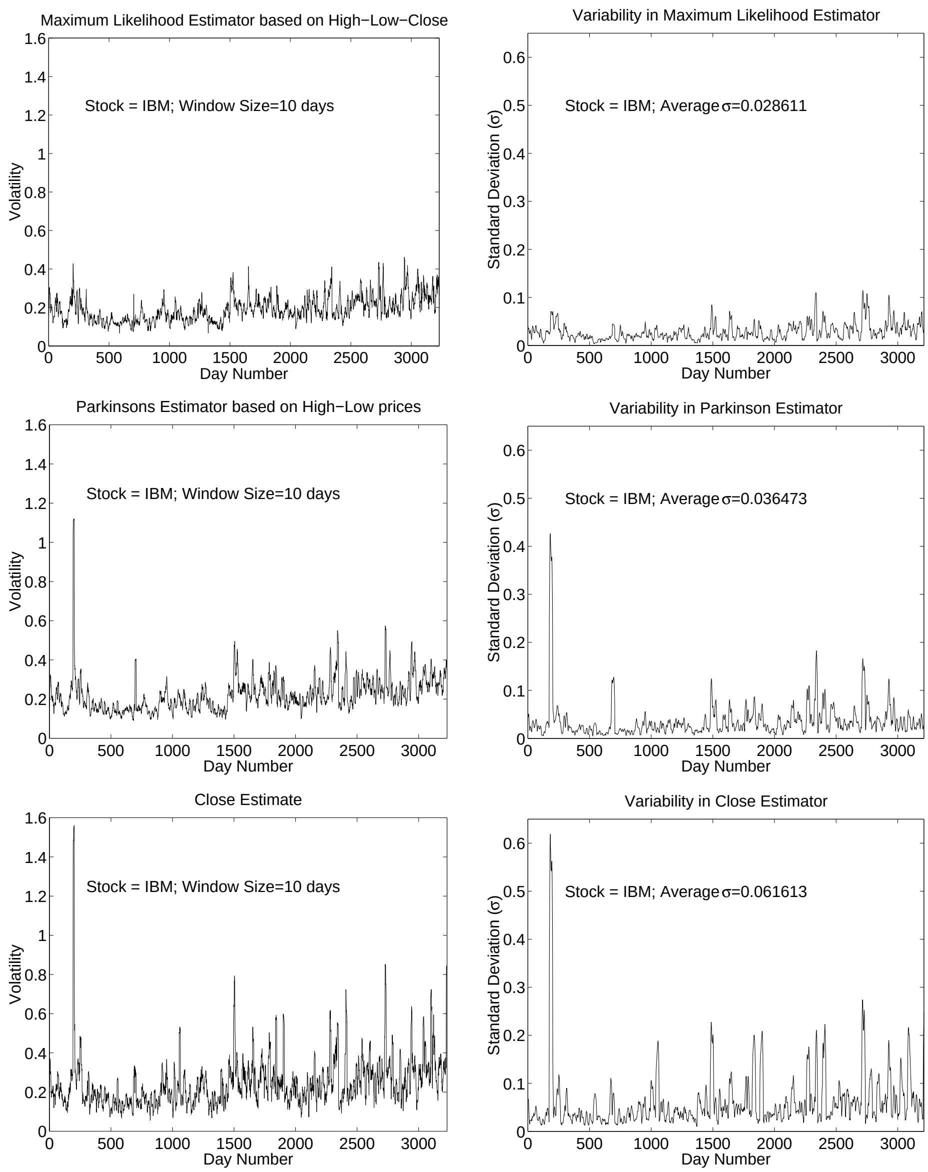

Figure 4: Volatility estimates on IBM's return series for various predictors.

In order to make the notation more concise, we will suppres the k, x, μ, σ, T dependence when referring to the above functions (keeping only the h_1, h_2 dependence), for example we will write $f(u)$ instead of $f(k,\mu,\sigma,u)$. The derivatives of $\tilde{C}(h_1,h_2)$ and $f(u)$ will be needed. We will use the usual subscript notation to denote the partial derivatives with respect to the arguments, for example, $\tilde{C}_{i,j}(h_1,h_2)$ is the i^{th} partial derivative with respect to the first argument and the j^{th} partial derivative with respect to the second argument. Using this notation, the joint density (6) is given by

$$p(h,l|x,\mu,\sigma,T) = - \left. F_{1,1}(h_1,h_2)\right|_{h1=l,h2=h} \tag{22}$$

We will need the partial derivatives $f_1(u)$, $f_2(u)$, $\tilde{C}_{0,1}(h_1,h_2)$, $\tilde{C}_{1,0}(h_1,h_2)$ and $\tilde{C}_{1,1}(h_1,h_2)$. Tedious but straightforward computations yield the following expressions.

$$f_1(u) = f(u)\left[\frac{T}{2\sigma^2 u}\left(\frac{g(u)}{u}-\mu^2\right)-\frac{\mu^2}{g(u)}\right] \tag{23}$$

$$f_2(u) = \frac{f_1(u)^2}{f(u)} + f(u)\left[\frac{T}{\sigma^2 u^2}\left(\mu^2-\frac{g(u)}{u}\right)+\frac{\mu^4}{g(u)^2}\right] \tag{24}$$

In order to write the derivatives of $\tilde{C}$ more compactly, introduce the function $A(h_1,h_2) = k\pi(x - h_1)/(h_2-h_1)$. The derivatives of A are then given by

$$A_{0,1}(h_1,h_2) = -\frac{A(h_1,h_2)}{(h_2-h_1)} \tag{25}$$

$$A_{1,0}(h_1,h_2) = A(h_1,h_2)\left(\frac{1}{h_2-h_1}-\frac{1}{x-h_1}\right) \tag{26}$$

$$A_{1,1}(h_1,h_2) = \frac{A(h_1,h_2)}{h_2-h_1}\left(\frac{1}{x-h_1}-\frac{2}{h_2-h_1}\right) \tag{27}$$

One then finds for the derivatives of $\tilde{C}$

$$\tilde{C}_{0,1}(h_1,h_2) = A_{0,1}\cot(A)\tilde{C} + (-1)^{k+1}\frac{\mu}{\sigma^2}\sin(A)exp\left(\frac{\mu}{\sigma^2}(h_2-x)\right) \tag{28}$$

$$\tilde{C}_{1,0}(h_1,h_2) = A_{1,0}\cot(A)\tilde{C} + \frac{\mu}{\sigma^2}\sin(A)exp\left(\frac{\mu}{\sigma^2}(h_1-x)\right) \tag{29}$$

$$\begin{aligned}\tilde{C}_{1,1}(h_1,h_2) &= A_{1,1}\cot(A)\tilde{C} + \frac{A_{1,0}A_{0,1}}{\sin^2(A)}\tilde{C} + A_{1,0}\cot(A)\tilde{C}_{0,1} \\ &\quad + \frac{\mu}{\sigma^2}A_{0,1}\cos(A)exp\left(\frac{\mu}{\sigma^2}(h_1-x)\right)\end{aligned} \tag{30}$$

where we have suppressed the (h_1,h_2) dependence of the A and $\tilde{C}$ functions on the right hand side. Finally the function $-F_{1,1}(h_1,h_2)$ can be obtained from the following expression

$$\begin{aligned}F_{1,1} = 2\sigma^4\pi\sum_{k=1}^{\infty}k\ &\Big[\tilde{C}_{1,1}f(u) + 2(h_2-h_1)f_1(u)(\tilde{C}_{1,0}-\tilde{C}_{0,1}) \\ &-4\tilde{C}f_2(u)(h_2-h_1)^2 - 2\tilde{C}f_1(u)\Big]\end{aligned} \tag{31}$$

References

[1] A Basso and P Pianca. A more informative estimation procedure for the parameters of a diffusion process. *Physica A*, 269:45–53, 1999.

[2] M. Dominé. First passage time distribution of a wiener process with drift concerning two elastic barriers. *Journal of Applied Probability*, 33:164–175, 1996.

[3] M. Garman and M. Klass. On the estimation of security price volatilities from historical data. *Journal of Business*, 53(1):67–78, 1980.

[4] M. Parkinson. The extreme value method for estimating the variance of the rate of return. *Journal of Business*, 53(1):61–65, 1980.

[5] L. Rogers. Volatility estimation with price quanta. *Mathematical Finance*, 8(3):277–290, July 1998.

[6] L. Rogers and S. Satchell. Estimating variance from high, low and closing prices. *Annals of Applied Probability*, 1(4):504–512, 1991.

[7] L. Rogers, S. Satchell, and Y. Yoon. Estimating the volatility of stock prices: a comparison of methods that use high and low prices. *Applied Financial Economics*, 4:241–247, 1994.

[8] L. A. Shepp. The joint density of the maximum and its location for a wiener process with drift. *Journal of Applied Probability*, 16:423–427, 1979.