

# ESTIMATION OF CREDIT AND INTEREST RATE RISKS FOR LOAN PORTFOLIOS

By

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## ABSTRACT

Banks need to estimate the risk of their loan portfolios, both for regulatory reasons and for purposes of ensuring the correct capital reserve in case of large losses. The two main risks of a loan portfolio are credit risk and interest rate risk. Most existing risk models compute credit and interest rate risks separately. We propose a model which estimates both credit and interest rate risks simultaneously. The model can also be used to separately estimate the two risks. Combined estimation of credit and interest rate risks is important, because in many real-world scenarios credit and interest rate risks are not independent. Our model calculates the probability distribution function (pdf) of portfolio losses for a given time horizon. Given the pdf of portfolio losses we can easily obtain various risk metrics, such as Value-at-Risk or Expected Shortfall. The model does not have a complete analytical solution for the loss pdf. We give an efficient approximation to the loss pdf based on a fast Monte-Carlo method. The running time of the algorithm is linear with respect to the number of loans. Also, we present a semi-analytical model for combined credit and interest rate risks. The solution for the semi-analytical approach is based on a numerical integration, which is performed efficiently using adaptive quadrature.

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# CHAPTER 1

## Introduction

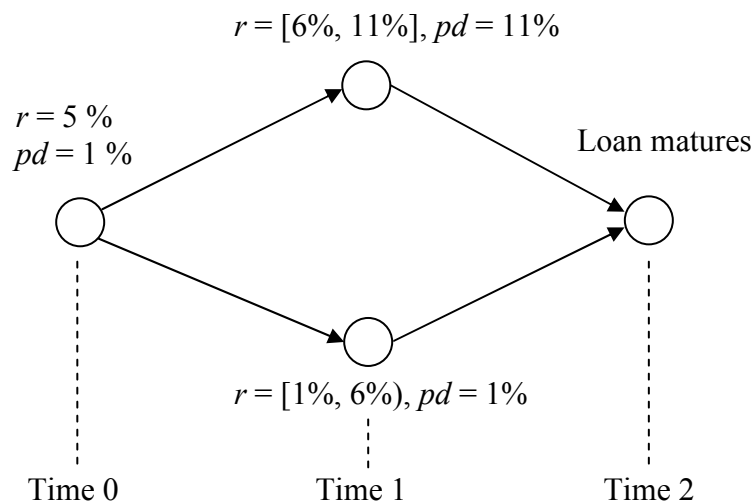
Accurate assessment of credit risk is very important for financial institutions nowadays. Market shocks can result in hundreds of billion dollar losses for US banks and broker-dealers, which shows that underestimation of risk is not cheap. Over-optimistic credit risk assessments lead to bankruptcies of financial institutions and could cause a systemic financial disruption. On the other hand over-pessimistic credit risk assessments lead to high cost of credit for companies and can depress economic activities, negatively affecting the whole economy. The accurate estimation of credit risk is very important for both financial institutions and economy in general.

One of the causes for underestimation of credit risk is the assumption that individual default events are independent, and pooling multiple credit instruments into banks' books or in structures like CDOs provides strong diversification making large losses extremely improbable. In reality, defaults are typically correlated through systemic factors, and when the market changes its regime, default losses could reach dangerous levels, threatening the solvency of a financial institution. Regulators recognize the need for more realistic risk management assessments and controls. Many currently used credit risk models do not adequately capture the dependency between individual defaults of different credit products.

Some existing closed-form models for credit risk rely heavily on various unrealistic assumptions making analytic derivations easier. Monte-Carlo simulations for credit risk are often very expensive computationally, which makes it prohibitive to model losses for a large credit portfolio. The interest rate plays a very important role in the performance of a credit portfolio. Current risk management frameworks usually compute interest rate risk separately from credit risk. That approach implicitly assumes the independence between credit risk and interest rate risk. However in the real world credit risk and interest rate risks are not independent. For example sharp interest rate increases could affect the ability of borrowers to repay variable

rate loans. We propose a model which takes into account default correlations among borrowers. Also we propose a model for interest rate risk and a framework which allows computing credit and interest rate risk simultaneously or separately. The model makes evaluation of credit and interest rate risk computationally feasible even for large portfolios consisting of hundreds of thousands of credit instruments. Our model evaluates credit risk for credit instruments within the credit portfolio which can have different maturities, different correlations to systemic factors, different losses given default, different amortization schedules and different probabilities of default. We tested our model using both simulated data and a real bank's loan portfolio. We find that the model makes accurate estimation of risk and successfully takes into account both credit and interest rate risk.

Let us introduce a simple example of a portfolio of one loan to show how credit and interest rate risks interact:



**Figure 1.1: Possible states of the loan**

At time 0 we have a loan, which pays interest continuously with cashflow rate of \$1 and matures at time 2 (two years after time 0). Initial probability of default ( $pd$ ) for the loan is 1%, the initial interest rate  $r$  is 5%. The loan defaults in time  $[0,1]$  with default probability  $pd = 1\%$ , at a uniformly distributed random time. At

time 1 new interest rate  $r$  is generated uniformly in the range  $[1\%, 11\%]$ , then if  $r < 6\%$  then  $pd$  becomes 1%, if  $r \leq 6\%$  then  $pd$  becomes 11%. The loan defaults in the range  $(1,2)$  with a default probability determined at time 1, and again, the time of default is a uniform random variable in  $(1,2)$ . If the loan doesn't default in the time interval  $[0, 2)$  then it matures at time 2. In the following table we list Value-at-risk (VaR) and Expected Shortfall values (ES) for different risk calculation methods. We see that the sum of credit and interest rate VaRs doesn't equal to combined credit and interest rate risk VaR, the same applies to expected shortfall values. We can see for the three different methods the relationship between VaRs and the relationship between Expected Shortfalls are not obvious.

**Table 1.1: Risk values for the one loan example**

Method	80% VaR	90% VaR	95% VaR	80% ES	90% ES	95% ES
Combined risk	0.0449	0.0602	0.0945	0.0841	0.1171	0.1621
Credit risk	0.0042	0.0334	0.1246	0.1095	0.1246	0.1695
Interest rate risk	0.0363	0.0516	0.0596	0.0519	0.0595	0.0635

If we change the dependence of  $pd$  on  $r$  to exactly opposite (if  $r \leq 6\%$  then  $pd$  becomes 1%, if  $r < 6\%$  then  $pd$  becomes 11%) then we will have different VaR and ES values:

**Table 1.2: Risk values for the one loan example, dependence of  $pd$  on  $r$  changed**

Method	80% VaR	90% VaR	95% VaR	80% ES	90% ES	95% ES
Combined risk	0.0479	0.0647	0.1165	0.0913	0.1266	0.1663
Credit risk	0.0042	0.0334	0.1246	0.1095	0.1246	0.1695
Interest rate risk	0.0363	0.0516	0.0596	0.0519	0.0595	0.0635

The VaR and ES values for separate credit risk and interest rate risk don't change, because those risks are not dependent on the correlation between credit

risk and the interest rate risk. Again, we see that there is no obvious relationship between the sum of credit and interest rate VaRs and combined credit and interest rate risk VaR. Expected Shortfall, which is the metric with better additive properties than VaR, also doesn't show even approximate additivity for the three different risk estimation methods.

## CHAPTER 2

### Historical review and related work

Researchers use different approaches for portfolio credit risk estimation. Most of the models for credit risk are factor based, and the most widely used factor probability distribution is the normal distribution. Some models introduce contagion effects, so that when an obligor defaults, that default affects other obligors. Vasicek [45] uses factor based model for homogeneous portfolios, with a single Gaussian market factor. Conditioned on the value of the market factor, the portfolio losses are approximated with the normal distribution. Andersen, Sidenius and Basu [8] use factor-based models with Gaussian market factors. They consider a discrete distribution of portfolio losses, so the loss is defined as an integer number of loss units. The authors propose to use Gaussian quadrature to calculate the unconditional loss distribution. Glasserman and Suchintabandit [34] propose a different model which is based on normal market factors, but the losses are modeled with a continuous distribution. For the correlation matrix of default variables they consider two cases - when correlations between default variables are strong and when they are weak. Numerical approximations are used to calculate the loss probability distribution. Hachet and Kuhn [20] present a model which has a single macro-economic factor and models contagion effects between companies. When one company defaults it can lead to negative or positive effects for other companies. The authors use numerical methods for loss estimation. Rosch and Winterfeldt [15] extend the standard factor model to include contagion effects. Gregory and Laurent [19] divide the credit portfolio into sectors. In each sector all obligors have identical exposure to factors. The authors propose to use some inversion technique such as FFT. Kuhn [30] extends the standard CreditMetrics model and calculates loss distributions for homogeneous portfolios. Berd, Engle and Voronov [1] introduce a factor model with a single market factor. They use a TARCH process to model the market factor in order to achieve desirable correlation surface properties. The model assumes that the credit portfolio is homogeneous. Grundke [18] models not only the credit risk for

a homogeneous portfolio, but also an interest rate risk using a factor model and a mean-reverting process for the interest rate. In summary there is much research on correlated, dependent credit defaults and some work on interest rate risk. Typically some simplifying assumptions are made such as homogeneous portfolios. In this light our research acknowledges that credit and interest rate risks are dependent. Our main contribution is a model for estimating combined credit and interest rate risks and efficient algorithms to implement our approach. Our models and algorithms do not rely on portfolios being homogeneous. We present a model which estimates credit risk for non-homogeneous portfolios, and has an efficient semi-analytical formulation. Further we extend that model to add interest rate risk, so the extended model estimates combined credit and interest rate risks for non-homogeneous portfolios.

We now review some of the most relevant work in detail

## 2.1 All Your Hedges in One Basket [8]

This work estimates the price of a CDO and the sensitivities of the CDO's price on underlying CDS spreads. Default times are modeled using a Student-t copula.  $Q$  is a risk-neutral probability measure and the joint default time distribution can be defined as:

$$Q(\tau \leq T) = t_{N,\nu}(t_{1,\nu}^{-1}(p_1(T_1)), \dots, t_{1,\nu}^{-1}(p_N(T_N))),$$

where  $t_{1,\nu}$  and  $t_{N,\nu}$  are the one and  $N$  - dimensional cumulative Student-t distribution functions with  $\nu$  degrees of freedom, respectively. For high degrees of freedom the Student-t copula approaches the Gaussian copula; However the Student-t copula gives fatter tails than the Gaussian copula.

$$Z = Y\sqrt{\nu/g},$$

where  $Y$  is  $N$  - dimensional standard Gaussian variable with correlation matrix  $\Sigma$  and  $g$  is a scalar chi-square distributed random variable with  $\nu$  degrees of freedom.

$Z$  follows  $N$ -dimensional Student-t distribution with correlation matrix  $\Sigma$ . In the paper loss defined not as a real number, but as an integer number of loss units, where one loss unit is a minimum loss amount. So the loss distribution in the paper is discrete. The general approach to obtain the loss distribution is recursive:

Suppose we know loss distribution  $p^K(l; t), l = 0, \dots, l_{max,K}$ , for a reference pool of some size  $K \geq 0$ . Now suppose we add another company with loss weight  $w_{k+1}$  and known default probability  $p_{k+1}(t)$ :

$$p^{k+1}(l; t) = p^k(l; t)(1 - p_{k+1}(t)) + p^k(l - w_{k+1})p_{k+1}(t), l = 0, \dots, l_{max,k} + w_{k+1}.$$

We can use this recursive function to build the loss distribution from a trivial base case. The paper assumes that  $Y$  can be represented as

$$Y = cX + \epsilon,$$

where  $X$  is an  $M$ -dimensional vector of independent standard Gaussian variables (factors),  $c$  is an  $N \times M$  loading matrix and  $\epsilon$  is an  $N$ -dimensional vector of independent zero-mean Gaussian variables with variances less than one. Authors use principal component analysis to calculate matrix  $c$  given  $\Sigma$  and  $M$ .

By conditioning on  $\Omega = (X, g)$  the authors arrive at:

$$p_k(t|\Omega) = \Phi \left( \frac{\sqrt{g} t_{1,\nu}^{-1}(p_k(t)) - c_k X}{\sqrt{1 - c_k c_k^T}} \right),$$

where  $c_k$  are the factor loadings and  $\Phi$  is the standard Gaussian cumulative distribution function. We can calculate the portfolio loss  $P(l; t|\Omega)$  using the recursive formula. Given  $\Omega$  all the companies in the portfolio are independent. Unconditional portfolio loss can be calculated using formula:

$$P(l; t) = \int_{R^M \times R_+} P(l; t|\Omega) q(\Omega) d\Omega.$$

The authors propose to use Gaussian quadrature to calculate the above integral. In addition to the portfolio losses authors calculate sensitivities of portfolios

value with respect to individual probabilities of default. The technique of conditional independence is also one we will use in our framework

**Summary.** This work is based on the Student-t copula and does not consider the interest rate. The discrete unit size is inconvenient but results in near exact (modulo quadrature) algorithm, albeit slow (quadratic in the number of loans).

## 2.2 Correlated Defaults and the Valuation of Defaultable Securities [47]

This paper models defaults using hazard rates. The author defines the total hazard accumulated by firm  $i$  by time  $t$  given  $n$  observed defaults as:

$$\psi_i(t|n) = \sum_{m=1}^n \Lambda_i(t_{k_m} - t_{k_{m-1}}|m-1) + \Lambda_i(t - t_{k_n}|n),$$

where  $\Lambda_i(s|m) = \int_{t_{k_m}}^{t_{k_m}+s} \lambda_i(u|m) du$  is the total hazard accumulated by firm  $i$  for a period of length  $s$  following  $m$ -th default, where  $\lambda_i$  is the default intensity of a loan  $i$ . It is assumed that there is no default between  $t_{k_m}$  and  $t$ . An inverse function that maps a unit exponential back into the original default time is defined as:

$$\Lambda^{-1}(x|n) = \inf\{s : \Lambda_i(s|n) \geq x\}, x \geq 0.$$

The following recursive procedure constructs a collection of random variables (default times)  $\tau = (\tau^1, \dots, \tau^I)$  :

1. Draw a collection of i.i.d unit exponentials  $E = (E^1, \dots, E^I)$ .
2. Let  $k_1 = \operatorname{argmin}\{\Lambda_i^{-1}(E^i|0)\}$  and let  $\tau^{k_1} = \Lambda_{k_1}^{-1}(E^{k_1}|0)$ .
3. Assume that the values of  $(\tau^{k_1}, \dots, \tau^{k_{m-1}})$  are already given, where  $m \geq 2$ . Define the set  $I_{m-1} = \{k_1, \dots, k_{m-1}\}$  and  $\overline{I_{m-1}}$  as the set of firms excluding  $I_{m-1}$ . Let  $k_m = \operatorname{argmin}_{i \in \overline{I_{m-1}}}\{\Lambda_i^{-1}(E^i - \psi_i(\tau^{k_{m-1}}|m-1)|m-1)\}$  and let  $\tau^{k_m} = \tau^{k_{m-1}} + \Lambda_{k_m}^{-1}(E^{k_m} - \psi_{k_m}(\tau^{k_{m-1}}|m-1)|m-1)$ .
4. If  $m = I$  then stop, otherwise increase  $m$  by 1 and go to step 3

For multi-obligor portfolio,  $\lambda_t^i = a_1 + a_2 1_{\{t \geq \tau_F\}}$ , where

$$\tau_F = \min(\tau^1, \dots, \tau^I)$$

is the first-to-default time. The marginal distribution of the default times can be obtained from this:

$$\tau_F = \min\left(\frac{E^1}{a}, \frac{E^2}{a}, \dots, \frac{E^I}{a}\right).$$

$$\tau^i = \frac{E^i - a_1 \tau_F}{a_1 + a_2} + \tau_F,$$

$$\tau_F = \min(E^1/a, \tau_F'),$$

where  $\tau_F' = \min(E^2, \dots, E^I)/a$ . The marginal density of  $\tau^1$  :

$$g_1(t_1) = \frac{(I-1)a_1(a_1+a_2)e^{-(a_1+a_2)t_1} - I a_1 a_2 e^{-I a_1 t_1}}{(I-1)a_1 - a_2}$$

**Summary.** This work introduces an intensity-based model and applies it for modeling defaults. The model does not consider the interest rate. The author doesn't give a solution for the model

### 2.3 Correlation Expansions for CDO Pricing [34]

There are  $M$  obligors in the portfolio; let  $Y_i$  be the default indicator for the  $i$ -th obligor,  $Y_i = (1 \text{ if } i\text{-th obligor defaults, } 0 \text{ otherwise})$ . Let  $p_i = P(Y_i = 1)$  be the marginal probability that the  $i$ -th obligor defaults.

Let  $c_i$  be the loss from the default of  $i$ -th obligor

Portfolio loss can be defined as:

$$L = c_1 Y_1 + c_2 Y_2 + \dots + C_m Y_m,$$

where  $Y_i = 1_{\{X_i > \nu_i\}}$ ,  $i = 1, 2, \dots, M$

The correlations in  $Y_i$  are introduced through correlations in the  $X_i$ .  $(X_1, X_2, \dots, X_m)$  are correlated  $N(0, 1)$  random variables. The "default boundary"  $\nu$  is chosen, so that  $p_i = 1 - \Phi(\nu_i)$ .

Correlations among  $X_i$  are introduced through the factors  $Z_1, Z_2, \dots, Z_d (d \ll M)$  :

$$X_i = a_{i1}Z_1 + a_{i2}Z_2 + \dots + a_{id}Z_d + b_i\epsilon_i.$$

Here  $\epsilon_i$  and  $Z_1, \dots, Z_d$  are independent  $N(0, 1)$  random variables,  $a_{ij}$  and  $b_i$  are constants such that  $a_{i1}^2 + \dots + a_{id}^2 + b_i^2 = 1$ , so  $X_i$  is  $N(0, 1)$ . Let  $C$  be the correlation matrix of  $X_i$  variables of obligors. The authors' goal is to approximate  $E(L - y)^+$ , where  $y$  is some fixed number. There are two general cases, the case of weak correlations in  $C$  and the case of strong correlations in  $C$ .

**Weak Correlations.** They obtain a parameterized matrix  $C_t$  by multiplying each element  $C_{ij}$  for  $i \neq j$  (i.e. diagonal elements were unchanged) by a constant  $t$ . Let  $E_t$  be the expectation under which  $X \sim N(0, C_t)$ ,  $t = 1$  (the original case). Setting  $t = 0$  corresponds to the case of independent obligors. It can be shown that  $E(L - y)^+$  is analytical in  $t$ , that is:  $E(L - y)^+ = \delta_0 + \delta_1 t + \delta_2 \frac{t^2}{2!} + \dots$ , where  $\delta_0, \delta_1, \dots$  are real scalars. Each  $\delta_n$  can be computed through a weighted finite sum of the form  $\sum_J w_j \tilde{E}_J(L - y)^+$ . The number of terms in the summation and the corresponding set of weights  $w_j$  depend on  $n$  and the number of factors  $d$  in the correlation structure. In each term  $\tilde{E}_J(L - y)^+$  is the expectation under which all obligors of  $L$  are independent with modified default probabilities which are perturbations of the original default probabilities. The magnitude of the perturbation is determined by a real variable  $s$ . As  $s$  approaches zero:

$$\sum_J w_j \tilde{E}_J(L - y)^+ \xrightarrow{s \downarrow 0} \delta_n.$$

The stronger the correlation among obligors, the more terms we need in the expansion. This approximation is appropriate when  $d > 2$  otherwise numerical integration may be a better option.

The same method can be used to approximate  $E_t[f(L)]$  where  $f$  is an arbitrary function of the loss portfolio.

**Strong Correlations** We define  $C_t = (1 - t)R + tC$ , where  $R$  is a reference correlation matrix for market factor  $Z$ , which has  $r$ -factor structure.  $r$  should be significantly smaller than  $d$ , for example  $r = 1$ . Let  $E_t$  be the expectation under  $C_t$ . The authors then derive analytic expressions of the form:

$$E_t(L - y)^+ = \Delta_0 + \Delta_1 t + \Delta_2 \frac{t^2}{2!} + \dots + \Delta_n \frac{t^n}{n!}$$

$$E_t[(L - y)^+ | Z = z] = \delta_0(z) + \delta_1(z)t + \delta_2(z) \frac{t^2}{2!} + \dots$$

For given  $z$ ,  $\delta_k(z)$  can be computed as described before. The  $\Delta_k$  are given by

$$\Delta_k = E[\delta_k(z)] = \int_{-\infty}^{+\infty} \delta_k(z) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.$$

Through numerical integration  $\Delta_0, \dots, \Delta_n$  can be computed to give the  $n$ -th order approximant for  $E_t(L - y)^+$ .

**Summary.** This work is based on multiple factors and does not consider the interest rate. The authors consider two different cases: the case of strong correlations among obligors and the case of weak correlations among obligors. The two cases have different solutions. The models use factor based correlations which is a similar approach to our single period credit risk only model.

## 2.4 Credit Contagion and Credit Risk [20]

Let  $W_{i,t}$  be the wealth or the default variable for a firm. If the wealth variable goes below zero, the firm defaults. So the wealth can be defined as:

$$W_{i,t} = \vartheta_i - \sum_{j=1}^N J_{ij} n_{j,t} - \eta_{i,t},$$

where  $\vartheta_i$  is the initial wealth of firm  $I$  at the beginning of risk horizon,  $J_{ij}$  quantifies the material impact on the wealth of firm  $i$ , that would be caused by default of firm  $j$ . Impact of default can be negative ( $J_{ij} > 0$ , cooperative) or positive ( $J_{ij} < 0$ , competitive) economic relation with  $i$ . Fluctuating contributions  $\eta_{i,t}$  are

zero-mean Gaussians. The model has a single macro-economic factor  $\eta_0$  (assumed to be constant over a risk horizon). So for each firm individual fluctuations can be defined as:

$$\eta_{i,t} = \sigma_i \left( \sqrt{p_i} \eta_0 + \sqrt{1-p_i} \xi_{i,t} \right),$$

where  $\xi_{i,t}$  are  $N(0,1)$  Gaussians,  $p_i$  quantify the correlations of the  $\eta_{i,t}$  created via the coupling to economy-wide fluctuations  $\eta_0$ , which are also  $N(0,1)$ . The model cannot have a drift. One year is split into twelve steps.  $W_{i,t} < 0$  is equivalent to  $\eta_{i,t} > \vartheta_i - \sum_{j=1}^N J_{ij} n_{j,t}$  when  $\sigma_i \equiv 1$  default occurs with probability  $\Phi \left( \sum_j J_{ij} - \vartheta_i \right)$ , if a company  $i$  has probability of default  $p_i$  in a given time unit then  $\vartheta_i = -\Phi^{-1}(p_i)$  the expected default probability  $p_{i|j}$  of firm  $i$ , given that only firm  $j$  has defaulted leads to the value  $J_{ij} = \Phi^{-1}(p_{i|j}) - \Phi^{-1}(p_i)$   
 $J_{ij} = c_{ij} \left[ \frac{J_0}{c} + \frac{J}{\sqrt{c}} x_{ij} \right]$ ,  $c_{ij} \in \{0, 1\}$   $c_{ij}$  shows presence or absence of interactions between different firms. So the authors define:

$$P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left( 1 - \frac{c}{N} \right) \delta_{c_{ij},0},$$

where  $x_{ij}$  are zero mean, unit variance random variables with finite moment and independent in pairs. Probability of default of firm  $i$  with average unconditional monthly default probability  $p_i$  is expressed as:

$$\langle n_{i,t+1} | n_{i,t} = 0 \rangle = \Phi \left( \frac{J_0 m_t + \sqrt{p_i} \eta_0 - \vartheta_i}{\sqrt{1-p_i + J^2 m_t}} \right),$$

where  $m_t = \frac{1}{N} \sum_j n_{j,t}$  is the fraction of firms within the economy that have defaulted up to time  $t$ . The fraction of firms that defaulted so far evolves according to:

$$m_{t+1} = m_t + \left\langle (1 - \langle \eta_t(\vartheta) \rangle) \Phi \left( \frac{J_0 m_t + \sqrt{p(\vartheta)} \eta_0 - \vartheta}{\sqrt{1-p(\vartheta) + J^2 m_t}} \right) \right\rangle,$$

where  $\langle \eta_t(\vartheta) \rangle$  is the time dependent monthly default rate of firms with  $\vartheta_i \approx \vartheta$

**Summary.** This work is based on a single factor and introduces contagion relationships between obligors. The model does not consider the interest rate. In order to use contagion effects impact parameters for obligors must be defined. The model assumes that the factor has zero drift

## 2.5 Estimating Credit Contagion in a Standard Factor Model [15]

For a loans in a credit portfolio we have default variables  $R_{i,t}$  for  $(i \in N_t, t = 1, \dots, T)$

$$R_{i,t} < c_{S(i)} \Leftrightarrow D_{i,t} = 1,$$

where  $D_{i,t} = 1$ , if borrower  $i$  defaults in period  $t$  and  $D_{i,t} = 0$  otherwise.  $S(i)$  denotes the segment borrower  $i$  belongs to  $S(\cdot) \rightarrow \{1, \dots, M\}$ . In each segment all borrowers are homogeneous with respect to parameters of their default processes (for example rating grades). The proces for the infecting firms is given as:

$$R_{i,t}^I = w_{S(i)} F_{S(i)t} + \sqrt{1 - w_{S(i)}^2} U_{i,t},$$

where  $F_{S(i)t} \sim N(0, 1)$ ,  $U_{it} \sim N(0, 1)$ ,  $F_{S(i)t}$  and  $U_{it}$  are independent random variables  $F_t = (F_{1t}, \dots, F_{Mt})$  is normally distributed with mean  $E(F_t) = 0$  and covariance matrix  $Cov(F_t)$ . The correlation between the creditworthiness indices of two firms is given by:

$$Corr(R_{it}, R_{jt}) = w_{S(i)} w_{S(j)} \sigma_{S(i), S(j)}.$$

Conditional probability of default can be derived as:

$$\pi_i(f_{S(i),t}) = P(R_{i,t} < c_{S(i)} | F_{S(i),t} = f_{S(i),t}) = \Phi \left( \frac{c_{S(i)} - w_{S(i)} f_{S(i),t}}{\sqrt{1 - w_{S(i)}^2}} \right),$$

$$\pi_i = \int_{-\infty}^{\infty} \Phi \left( \frac{c_{S(i)} - w_{S(i)} f_{S(i),t}}{\sqrt{1 - w_{S(i)}^2}} \right) d\Phi(f_{S(i),t}) = P(R_{i,t} < c_{S(i)}) = \Phi(c_{S(i)}).$$

Default variable with default contagion can be expressed as:

$$R_{i,t}^I = w_{S(i)} F_{S(i),t} + \sqrt{1 - w_{S(i)}^2} U_{i,t} + \beta \frac{\sum_{s \in I_t[m, K(j)]} D_{s,t}^I}{I_{K(j),t}} = w_{S(i)} F_{S(i),t} + \sqrt{1 - w_{S(i)}^2} U_{i,t} + \beta \frac{D_{K(j),t}^I}{I_{K(j),t}},$$

where  $K(j)$  denotes the contagion segment borrower  $j$  belongs to, e.g. and industry sector ( $K(\cdot) : \{1, \dots, N_t\} \rightarrow \{1, \dots, K\}$ )  $I_t[m, K(j)] = \{i | S(i) = m, K(i) = K(j), \text{infecting}\}$  - set of infecting firms belonging both to rating grade  $m$  and industry segment  $K(j)$  at time  $t$ , and  $I_{K(j),t}$  be their number  $D_{K(j),t}^I = \sum_{s \in I_t[m, K(j)]} D_{s,t}^I$  is the number of defaulting infecting firms in segment  $K(j)$ . Let  $\beta$  be a unknown coefficient which measures the impact of contagion on the default probability. The effect of contagion on firm  $j$  is then  $\beta$  times the default rate of the infecting firms within industry  $K(j)$ . Conditional on the risk factors and the number of defaulting infectors  $D_{K(j),t}^I = d_{K(j),t}^I$  one obtains the conditional probability

$$\pi_j^C(f_{S(i),t}, d_{K(j),t}^I(f_t)) = \Phi\left(\frac{c_{S(j)} - w_{S(j)} f_{S(j),t} - \beta \frac{d_{K(j),t}^I(f_t)}{I_{K(j),t}}}{\sqrt{1 - w_{S(j)}^2}}\right).$$

**Summary.** This work is factor-based and introduces contagion relationships between obligors. The model does not consider the interest rate. A loan portfolio is divided into sectors and in each sector all the loans are homogeneous. In order to use contagion effects the impact parameters for obligors must be defined. Despite all this machinery the authors do not provide a formula for the unconditional portfolio loss.

## 2.6 In the Core of Correlation [19]

The first part of the paper describes dealing with correlation bump, which we don't include into this review.

Authors divide all the loans into groups (e.g. industry sectors) and then create a model using intra-group correlations and inter-group correlations. A loan's default

variable  $V_i$  can be expressed as:

$$V_i = \rho_{k(i)} W_{k(i)} + \sqrt{1 - \rho_{k(i)}^2} \bar{V}_i,$$

where  $\rho_{k(i)}$  is the correlation of the default variable to a group  $\bar{V}_i$ ,  $W_{k(i)}$  are independent standard Gaussian variables, and  $k(i)$  is a sector of  $i$ . For simplicity all obligors in the sector have identical exposure to factor  $W$ , so  $W_j$  can be expressed as:

$$W_j = \lambda_j W + \sqrt{1 - \lambda_j^2} \bar{W}_j,$$

where  $W$ ,  $\bar{W}_j$ ,  $\bar{V}_i$  are independent standard Gaussian variables. So we can define  $V_i$  as:

$$V_i = \rho_{k(i)} \lambda_{k_i} W + \rho_{k(i)} \sqrt{1 - \lambda_{k(i)}^2} \bar{W}_{k(i)} + \sqrt{1 - \rho_{k(i)}^2} \bar{V}_i.$$

For two names in the same sector the correlation is equal to  $\rho_{k(i)}^2$ , for two names in different sectors  $k(i), k(j)$ ,  $k(i) \neq k(j)$ , the correlation is equal to  $\rho_{k(i)} \rho_{k(j)} \lambda_{k(i)} \lambda_{k(j)}$  which is in the range  $[0, 1]$  and inter-sector correlation coefficients will be smaller than intra-sector correlation coefficients. The most obvious correlation structure we can deal with this way is where all correlations are equal to  $\gamma$ , but with no limitations on the number of sectors. This is calibrated through  $\rho_i = \sqrt{\beta_i}$ , and  $\lambda_i = \sqrt{\frac{\gamma}{\beta_i}}$  for  $i = 1, \dots, m$ , where  $m$  is the number of sectors. So the accumulated loss at time  $t$  is defined as:

$$L(t) = \sum_{i=1}^n M_i N_i(t),$$

where  $M_i$  is the loss given default for obligor  $i$ ,  $N_i(t)$  is the default indicator variable for obligor  $i$  at time  $t$ . Probability generating function for  $L(t)$  can be expressed as:

$$\psi_{L(t)}(u) = E [u^{L(t)}] = E [E [u^{L(t)} | W]].$$

Losses are separated over different sectors  $L_j(t) = \sum_{i, k(i)=j} M_i N_i(t)$  and

$L(t) = \sum_j L_j(t)$ . Given  $W$  and  $L_j(t)$  are independent we can derive:

$$\psi_{L(t)}(u) = E \left[ \prod_j E [u^{L_j(t)} | W] \right],$$

$$E [u^{L_j(t)} | W] = E [E [u^{L_j(t)} | W, \overline{W}_j] | W].$$

For  $i$ ,  $k(i) = j$ ,  $N_i(t)$  are independent conditionally on  $W, \overline{W}_j$

$$E [u^{L_j(t)} | W, \overline{W}_j] | W = \prod_{i, k(i)=j} E [u^{M_i, N_i(t)} | W, \overline{W}_j] = \prod_{i, k(i)=j} (q_t^{i|W_j} + p_t^{i|W_j} u^{M_i}).$$

With  $p_t^{i|W_j} = \Phi \left( \frac{-\rho_i W_j + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}} \right)$  and  $q^{i|W_j} = 1 - p_t^{i|W_j}$ , since  $W_j$  is known whenever  $W, \overline{W}_j$  are known we have:

$$E [u^{L_j(t)} | W] = E \left[ \prod_{i, k(i)=j} (q_t^{i|W_j} + p_t^{i|W_j} u^{M_i}) | W \right],$$

$$\psi_{L(t)}(u) = E \left[ \prod_j E \left[ \prod_{i, k(i)=j} (q_t^{i|W_j} + p_t^{i|W_j} u^{M_i}) | W \right] \right].$$

It can be seen that the burden of computing the probability generating function is similar to a two-factor Gaussian structure. The distribution  $L(t)$  is obtained by some inversion technique such as FFT.

The third part of the paper describes the model where default dates and recovery rates are dependent, which we don't include into this review.

**Summary.** This work is factor-based and does not consider the interest rate. A loan portfolio is divided into groups (e.g. industry sectors) and the intra-sector correlations are exactly the same. The authors don't give a solution to the model and propose to use Fast Fourier Transformation for that purpose.

## 2.7 Tails of Credit Default Portfolios [30]

We consider a homogeneous portfolio of  $m$  bonds, then the fraction of defaulted loans  $L^{(m)}$  can be expressed as:

$$L^{(m)} = \frac{1}{m} \sum_{j=1}^m L_j,$$

where  $L_j = 1$  indicates the default of the credit of company  $j$ . Each bond is characterized by the vector  $(S_j, s)$  where  $S_j$  is a default variable and  $s$  is a default threshold. Default occurs when  $S_j < s$ . So the default variable can be expressed as:  $S_j = W \cdot s \cdot (X, Y_j)$ , where  $W > 0$ ,  $X \in R$  and  $(Y_j)(j \in N)$  is an iid sequence of real random variables. The  $Y_j$  are company-specific risk factors,  $X$  is a common risk factor,  $W$  is a global risk factor.

$$S_j = W(aX + bY_j),$$

where  $a, b > 0$  and  $W > 0$ ,  $X, Y_j \in R$

The CreditMetrics model corresponds to  $W = 1$ ,  $X, Y_j \sim N(0, 1)$ , and  $a = \sqrt{\rho}$ ,  $b = \sqrt{1 - \rho}$ , for some  $\rho \in (0, 1)$  we have:

$$L_j = 1_{\{S_j < s\}} = 1_{\{W_s \cdot (X, Y_j) < s\}},$$

$$L^m = \frac{1}{m} \sum_{j=1}^m 1_{\{W_s \cdot (X, Y_j) < s\}},$$

$$\lim_{m \rightarrow \infty} L^{(m)} = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=1}^m L_j = E(L_1 | W, X) = P(S_1 < s | W, X),$$

$$L = F_Y(y \cdot (s | W, X)),$$

$$p_{loss} = P(L_j = 1), 0 \leq \sqrt{Var(L)} \leq \sqrt{p_{loss}(1 - p_{loss})},$$

So we have the relation:

$$L\left(\frac{1}{1-q}\right) \sim C_{\mu_x} \mu_X^{(\mu_x-1)/2} \int_0^\infty \left( -\frac{s}{aw} (1-q)^{1/\nu_Y} + \frac{b}{a} C_{\nu_Y}^{1/\nu_Y} \nu^{(\nu_Y-1)/2\nu_Y} \right)^{-\nu_X} dF_w(w).$$

**Summary.** This work is factor-based and models only homogeneous portfolios using a single factor. The model does not consider the interest rate.

## 2.8 The Underlying Dynamics of Credit Correlations [1]

We have a loan portfolio. For loans in the portfolio  $\bar{R}_i \in [0, 1]$  is constant recovery rate. Total loss  $L_M$  can be expressed as:

$$L_M = \sum_{i=1}^M l_i 1_{\{\tau_i \leq T\}} = \sum_{i=1}^M l_i Y_i,$$

where  $l_i$  is the loss given default for a loan  $i$  and  $Y_i$  is a default indicator variable for a loan  $i$ . Expected value of  $L_M$  can be derived as:

$$E(L_M) = \sum_{i=1}^M l_i E(Y_i) = \sum_{i=1}^M l_i p_i,$$

where  $p_i$  is the probability of default for loan  $i$ . The default variable  $R_i$  can be derived as:

$$R_i = bR_m + \sqrt{1 - b^2} E_i,$$

where  $R_m$  is the market factor  $0 \leq b \leq 1$ ,

$Y_i = 1 \Leftrightarrow R_i \leq d_i$ , for  $i = 1, \dots, M$

The total loss of the portfolio given the value of the market factor can be derived as:

$$L^G = (1 - \bar{R})N\Phi\left(\frac{\Phi^{-1}(\rho) - \sqrt{\rho}R_m}{\sqrt{1 - \rho}}\right),$$

where  $\bar{R}$  - recovery rate,  $N$  - total portfolio exposure,  $\rho$  - correlation of a loan to the market factor,  $R_m$  is the market factor.

The authors choose TARARCH to model the market factor with good correlation surface properties, such as correlation skew for long time-periods (5 or 10 year horizon). Geometric Brownian Motion modeling leads to the normal distribution and is not realistic (does not match excess kurtosis etc). The TARARCH(1,1) model (asymmetric

threshold GARCH) can be expressed as:

$$r_t = \sigma_t \varepsilon_t,$$

$$\sigma_t^2 = w + \alpha r_{t-1}^2 + \alpha_d r_{t-1}^2 1_{\{r_{t-1} \leq 0\}} + \beta \sigma_{t-1}^2.$$

Here  $\{\varepsilon_t\}$  are *iid*, have zero mean, variance normalized to 1, finite skewness  $s_\varepsilon$  and finite kurtosis  $k_\varepsilon$ ,  $w > 0$ ,  $\alpha$ ,  $\alpha_d$ ,  $\beta$  are non-negative parameters. The persistence of volatility is governed by the parameter

$$\zeta \equiv E(\beta + \alpha \varepsilon_t^2 + \alpha_d \varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}) = \beta + \alpha + \alpha_d v_\varepsilon^d,$$

$$v_\varepsilon^d \equiv E(\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}),$$

$$k_\varepsilon^d \equiv E(\varepsilon_t^4 1_{\{\varepsilon_t \leq 0\}}).$$

The authors achieve better replication of correlation surface properties using TARARCH, than using standard large homogeneous portfolio model.

**Summary.** This work models homogeneous portfolios only, when the market factor is following the TARARCH process in order to achieve good correlation surface properties. The model does not consider the interest rate. The authors do not give an analytical solution and propose to use Monte-Carlo to obtain the loss distribution.

## 2.9 Integrating Interest Rate Risk in Credit Portfolio Models [18]

The credit portfolio consists of  $N$  coupon bonds with identical face value  $F$ , maturity date  $T$ , coupon  $c$  and coupon dates  $H \leq t_1 \leq \dots \leq t_m = T$ . Let  $P$  be the real world probability measure with asset return defined as:

$$R_j = w_1 Z + w_2 X_r + w_3 \varepsilon_j,$$

where  $j \in 1, \dots, N$ ,  $Z, X_r, \varepsilon_j \sim N(0, 1)$ ,  $Cov(Z, X_r) = 0$ ,  $Cov(Z, \varepsilon_j) = 0$ ,  $Cov(X_r, \varepsilon_j) = 0$

$Z$  is a market factor,  $\varepsilon_j$  is a firm specific risk variable,  $X_r$  is a stochastic factor driving the term structure of risk-free interest rates, so we have:

$$\begin{aligned} dr(t) &= \kappa(\theta - r(t))dt + \sigma dW(t), \\ r(t|F_s) &\sim N(\mu_r(s, t), \sigma_r^2(s, t)), \\ \mu_r(s, t) &= \theta + (r(s) - \theta)e^{-\kappa(t-s)}, \\ \sigma_r^2(s, t) &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-s)}), \\ r(H|F_0) &= \theta + (r(0) - \theta)e^{-\kappa H} + \sqrt{\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa H})}X_r, \end{aligned}$$

$r(H)$  is the risk-free interest rate at the risk horizon, since  $X \sim N(0, 1)$ , it's possible that  $r(t)$  can be negative with small probability:

$$p(r(t), t, T) = F e^{\left(\frac{1}{\kappa}(1 - e^{-\kappa(T-t)})(R(\infty) - r(t)) - (T-t)R(\infty) - \frac{\sigma^2}{4\kappa^3}(1 - e^{-\kappa(T-t)})^2\right)},$$

$$p(X_r, t, T) = F e^{\left(\frac{1}{\kappa}(1 - e^{-\kappa(T-t)})(R(\infty) - (\theta + (r(0) - \theta)e^{-\kappa t} + \sqrt{\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})}X_r)) - (T-t)R(\infty) - \frac{\sigma^2}{4\kappa^3}(1 - e^{-\kappa(T-t)})^2\right)}.$$

$R(\infty) = \theta + \lambda \frac{\sigma}{\kappa} - \frac{1}{2} \frac{\sigma^2}{\kappa^2}$  is the return of a risk-free zero coupon bond with infinite time to maturity, - constant market price of interest rate risk. Sensitivities to risk factors  $w_1, w_2, w_3$  are identical to all debtors, which implies identical correlation  $\rho$  between all pairs of asset returns. Without loss of generalization, the variance of the asset returns  $R_j$  can be normalized to one, so we have:

$$Var(R_j) = w_1^2 + w_2^2 + w_3^2 = 1,$$

$$\rho = Corr(R_i, R_j) = w_1^2 + w_2^2,$$

$$w_3 = \sqrt{1 - \rho}.$$

Default happens if  $R_j < \alpha$ ,  $q = P(R_j \leq \alpha) = \Phi(\alpha)$

$$\begin{aligned} P(r_j \leq \alpha | Z = z, X_r = x_r) &= P(\varepsilon_j \leq \frac{\alpha - w_1 Z - w X_r}{\sqrt{1 - \rho}} | Z = z, X_r = x_r) = \\ &= \Phi\left(\frac{\alpha - w_1 z - w x_r}{\sqrt{1 - \rho}}\right) = q(z, x_r). \end{aligned}$$

Price of a coupon bond with face value  $F$ , maturity date  $T$ , coupon  $c$  and coupon dates  $H \leq t_1 \leq \dots \leq t_M = T$  at the risk horizon whose issuer  $j$  has not yet defaulted ( $nd$ ) up to this time is defined as:

$$\nu_j^{nd}(X_r, H, T, c) = \sum_{m=1}^M c e^{-(R(X_r, H, t_m) + S(0, H, t_m))(t_m - H)} + F e^{-(R(X_r, H, T) + S(0, H, T))(T - H)},$$

$R(X_r, H, t_m) = -(t_m - H)^{-1} \ln(p(X_r, H, t_m)/F)$  is the stochastic risk-free discount factor for the time interval  $[H, t_m]$ ;  $S(0, H, t_m)$  is the forward credit spread for the time interval  $[H, t_m]$  observed at  $t = 0$ ;  $\nu_j^d = \delta F$  is the price of defaulted loan;  $\delta$  is the recovery rate. The value of the portfolio is expressed as:

$$\Pi(H) = [D\nu^d + (1-D)\nu^{nd}(X_r, H, T, c)]N = [\nu^{nd}(X_r, H, T, c) + D(\nu^{nd}(X_r, H, T, c) - \delta F)]N.$$

If we assume that the portfolio of corporate coupon bonds is sufficiently large so that the fraction  $D$  of defaulting debtors given a certain realization of the risk factors  $Z = z$ ,  $X_r = x_r$  is adequately approximated by the conditional default probability  $q(z, x_r)$ , we get with the law of iterated expectations for the probability distribution of the credit portfolio value  $\Pi(H)$  at the risk horizon:

$$\begin{aligned} P(\Pi(H) \leq \pi) &= P([\nu^{nd}(X_r, H, T, c) - D(\nu^{nd}(X_r, H, T, c) - \delta F)]N \leq \pi) = \\ &= E^p[P([\nu^{nd}(X_r, H, T, c) - D(\nu^{nd}(X_r, H, T, c) - \delta F)]N \leq \pi) | Z, X_r] = \\ &= E^p[P([\nu^{nd}(X_r, H, T, c) - q(Z, X_r)(\nu^{nd}(X_r, H, T, c) - \delta F)]N \leq \pi) | Z, X_r]. \end{aligned}$$

**Summary.** This work models homogeneous portfolios only and considers the interest rate. Authors assume that future credit spreads are deterministic

## CHAPTER 3

### The Credit Risk Model

We will first develop the model in which a static interest rate term structure is assumed. Then we will extend the model in Chapter 6 to add dependent interest rate dynamics.

#### 3.1 The Basic Model

Let  $P(t)$  denote the zero coupon bond term structure, which could equivalently be specified by forward rate term structure or yield curves. Let  $C_i(t)$  denote the cashflow rate for a deterministic deal  $i$ . The value of the deal can then be computed as

$$V_i = \int_0^{\infty} dt C_i(t)P(t),$$

Because the deal is deterministic we will make the following simplifying, practically valid assumption that  $C(t) > 0$  on  $(0, \infty)$ . For loans and credit products such as CDOs, cashflow rates are always non-negative. This assumption corresponds to loan type deals in which an initial loan at  $t = 0$  is followed by payments.

Deal  $i$  has an associated survival probability curve,  $s_i(t)$ , which we assume is time homogeneous and memoryless. Thus,

$$s_i(t) = e^{-\alpha_i t}.$$

A corresponding default probability density  $f_D(t)$ , equal to the probability density of default in  $(t, dt)$  can be computed as follows.

$$\begin{aligned} f_{D_i}(t) &= \lim_{\Delta t \rightarrow 0} \mathbf{Pr}[\text{survive to } t] \cdot (1 - \mathbf{Pr}[\text{survive to } t + \Delta t | \text{survive to } t]), \\ &= \lim_{\Delta t \rightarrow 0} s_i(t)(1 - \mathbf{Pr}[\text{survive to } t | \text{survive to } t + \Delta t] \cdot s_i(t + \Delta t)/s_i(t)), \\ &= \lim_{\Delta t \rightarrow 0} s_i(t)(1 - s_i(t + \Delta t)/s_i(t)), \\ &= \alpha_i e^{-\alpha_i t}. \end{aligned}$$

We will assume that these are the risk neutral default probabilities, so that we can value loans by taking expectations. Users of the model may need to adjust the Loss Given Default (LGD) parameters for loans in order to satisfy the risk neutral assumption. Suppose we are interested in the time horizon  $T$ , typically 1 year, and let  $X_i$  be a random variable which is the value of the deal at time  $T$  in present dollars.

In the event of default, let  $\rho_i$  be the (deterministic) recovery rate,  $0 \leq \rho_i \leq 1$ . An additional complexity would arise from making the recovery rate stochastic, and the marginal gain is not quite clear. The recovery rate  $\rho_i$  specifies the fraction of the remaining future cashflows which can be recovered in the event of a default. Alternatively, this could be the fraction of the outstanding capital which could be recovered – either interpretation is essentially equivalent.

### 3.2 A Single Deal With Term Structure

Suppose we are interested in the expected value and variance of the loan's value distribution at time  $T$ . To compute the value of a loan at time  $T$  we use the expected discounted cashflows, where the expectation is according to the assumed risk neutral default density

**Theorem 3.2.1** *The two central moments of loan's value distribution at time  $T$  are*

$$\begin{aligned} \mathbf{E}[X_i] &= X_{i|\infty} - (1 - \rho_i) \int_0^\infty dz (1 - e^{-\alpha_i z}) C_i(z) P(z), \\ \mathbf{var}[X_i] &= (1 - \rho_i)^2 \left[ 2 \int_0^\infty dz C_i(z) P(z) e^{-\alpha_i z} \int_0^z dw C_i(w) P(w) \right] \\ &\quad - (1 - \rho_i)^2 \left( \int_0^\infty dz C_i(z) P(z) e^{-\alpha_i z} \right)^2. \end{aligned}$$

*The proof is given in the appendix*

### 3.2.1 Examples

We now consider two basic examples, the uniformly amortized loan and the uniformly amortized bullet loan. We will always assume that the initial loan amount (principal  $P$ ) which is a negative cashflow has already been taken into account in the portfolio. Typically this will be based on money borrowed at a lower interest rate, and will represent an outgoing cash stream with no default risk and hence is easy to value.

In the simplest interest rate model, we assume a fixed interest rate, hence  $P(t) = e^{-rt}$ . This also assumes no interest rate dynamics and hence no interest rate risk. With interest rate dynamics, all of this formulation must change.

It will be convenient to define  $\mathcal{Z}(T; \alpha) = \int_0^T dt (1 - e^{-\alpha t})P(t)$ , which for the simple case of fixed interest rate evaluates to,

$$\mathcal{Z}(T; \alpha) = \frac{\alpha}{\alpha + r} - e^{-rT} \left( \frac{1}{r} - \frac{e^{-\alpha T}}{\alpha + r} \right).$$

#### Uniformly Amortized Loan

$$C_i(t) = \begin{cases} c_i & t \in [0, \tau_i], \\ 0 & \text{otherwise.} \end{cases}$$

#### Uniformly Amortized Bullet Loan

$$C_i(t) = \begin{cases} c_i + P \cdot \delta(t - \tau_i) & t \in [0, \tau_i], \\ 0 & \text{otherwise,} \end{cases}$$

where  $P$  is a bullet lump payment at maturity.

## 3.3 Correlated Defaults - Multiple Deals

### 3.3.1 Single Period

Consider  $n$  deals. To each deal  $i$ , associate a default variable  $d_i$  which is driven by  $K$  factors  $f_1, \dots, f_K$  together with its own stochastic component  $\epsilon_i$ . The default variable represents the financial health of the borrower. In our model the default

variable starts at 0 which corresponds to a simple recalibration of the initial financial health. The default variable can go up or down. If the default variable goes down and hits the default threshold which is always negative, the borrower defaults. Each loan has its own default threshold which corresponds to the loan's probability of default. For borrowers which are public companies, the default variables should be functions of the market capitalizations of the companies. If the market capitalization of the company approaches zero, the company goes into bankruptcy and defaults. In our model 0 would map to the initial market capitalization of the company and the value of the default threshold would correspond to zero market capitalization. If market capitalizations of public companies are good proxies for financial health, then in this case the default variables are observable through stock prices. For private companies the situation is less transparent. For private companies the default variable represents the difference between assets and liabilities. When liabilities are larger than assets we have the default event. For individual customers the default variable represents the net worth of a customer. The customer defaults when his or her net worth goes negative. On the one hand, there is the measurability of the default variable, which is needed to specify some of the quantities in the model - in particular the correlation of the default variable to the market factor. In practice this may not be possible and so one uses a reasonable value for these correlations, for example 19% suggested by MKMV. On the theoretical side, the postulation of the existence of such a default variable is innocuous, since it merely postulates the existence of a measure of financial worth, which drives default. This is a good approximation to reality.

Thus,

$$d_i = \sum_{k=1}^K \lambda_{ik} f_k + \beta_i \epsilon_i,$$

where we assume that the set of random variables  $\{f_1, \dots, f_K, \epsilon_1, \dots, \epsilon_n\}$  are independent with zero mean and unit variance. Intuitively,  $d_i$  is some form of normalized asset value of the underlying obligor. The factors  $f_k$  are common to all deals. They could for example represent obligor, sector, market sentiment, geography, etc. We

will assume a normalization condition

$$\beta_i^2 + \sum_{k=0}^K \lambda_{ik}^2 = 1,$$

for all  $i$ , which corresponds to each  $d_i$  having zero mean and unit variance. All these assumptions on the scales of the parameters are innocuous and amount to recalibration of the default thresholds. These thresholds in turn will be determined from the probabilities of default, which are user supplied inputs and will not be affected by such rescaling. The correlation  $\rho_{ij}$  between  $d_i$  and  $d_j$  is given by

$$\rho_{ij} = \beta_i \beta_j \delta_{ij} + \sum_{k=0}^K \lambda_{ik} \lambda_{jk},$$

where  $\delta_{ij}$  is the Kroeneker delta. Associated with each deal  $d_i$  is a default event  $D_i$ , which corresponds to a threshold  $t_i$  such that  $D_i = 1$  if  $d_i \leq t_i$  and  $D_i = 0$  otherwise. We can calculate the value of the default threshold using the following approach: we look at the relationship between probability of default  $pd$ , drift  $\mu$ , standard deviation  $\sigma$ , credit risk horizon  $T$  and default threshold  $l$ :

$$pd = \Phi\left(-\frac{\mu T + l}{\sigma \sqrt{T}}\right) + e^{-\frac{2\mu l}{\sigma^2}} \Phi\left(\frac{\mu T - l}{\sigma \sqrt{T}}\right).$$

It's not easy to transform this function to express  $l$  analytically as a function of other variables. We will use a simple root finding algorithm based on bisection to solve this implicit equation for  $l$  given  $\mu, \sigma, T, pd$ .

We define the random variable  $D = \sum_i D_i$  which is the total number of defaults. In what follows, we will simplify to the single factor model, because we have found practically that a single factor usually suffices. Besides, it is already challenging with just one factor. Thus,

$$d_i = \lambda_i f + \sqrt{1 - \lambda_i^2} \epsilon_i.$$

For every loan, we need to estimate the correlation to the market factor  $\lambda_i$ . For public companies the market capitalization represents the default variable. So

we can estimate the correlation to the market factor by calculating the correlation of the historical stock prices to historical values of the market factor in the case that the market factor is observable. In the case of a private company it's harder to estimate the correlation to the market factor, because we don't have historical stock prices. However we can use various approaches to estimate the correlation to the market factor: for example, we can take the market sub-index of publicly traded companies in the same industry, calculate the historical correlation of the industry sub-index to the market and then adjust the correlation taking into account the differences between the private company and publicly traded companies in the same industry. For individual customers the correlation to the market factor will depend on the relationship between the ability of a borrower to repay the loan and observable economic statistics such as unemployment rate, relevant housing prices (housing prices affect the individual's net worth in a major way) and possibly some other relevant statistics with available historical data. The specific models for the estimation of correlation of individual's default variable to the market factor are outside of the scope of this work, but such models are important and can be constructed. Our model takes the correlations between loans and the market factor as inputs.

We note that conditioned upon  $f$ , the defaults are independent. To see this, note that conditioned on  $f$ ,  $d_i$  is a function of  $\epsilon_i$  and default only depends on  $d_i$ , hence only on  $\epsilon_i$ , and the  $\epsilon_i$  are independent.

We will illustrate with the following simple single period model, which is already pretty realistic. Let the notional of deal  $i$  be  $X_i$ , and assume that there is some profit spread  $\gamma_i$ , in the event of no default. That is, we borrow at an interest rate which is approximately  $\gamma_i$  less than the interest rate at which we lend. In the event of default, we have a loss given default fraction (LGD)  $\delta_i$  (i.e., we recover  $(1 - \delta_i)$  of the loaned notional  $X_i$ ). In this case, the value  $v_i$

$$\text{Value } v_i = \begin{cases} \gamma_i X_i & D_i = 0, \\ -\delta_i X_i & D_i = 1. \end{cases}$$

Conditioning on  $f$ , lets define  $p_i(f) = \mathbf{Pr}[D_i = 1|f]$ , which is the probability of default on deal  $i$  given  $f$ . Let  $F_i$  be the distribution function for  $\epsilon_i$ , and let  $F$  be the distribution function for  $f$ . Then default occurs if and only if  $d_i \leq t_i$ , i.e. iff  $\lambda_i f + \sqrt{1 - \lambda_i^2} \epsilon_i \leq t_i$ , i.e. iff  $\epsilon_i \leq (t_i - \lambda_i f) / \sqrt{1 - \lambda_i^2}$ , and so we have that

$$p_i(f) = \mathbf{Pr}[D_i = 1|f] = F_i \left( \frac{t_i - \lambda_i f}{\sqrt{1 - \lambda_i^2}} \right).$$

Let  $\mu_i(f) = \mathbf{E}[v_i|f]$  be the expected value (with respect to default or no-default), conditioned on  $f$ , and similarly let  $\sigma_i^2(f) = \mathbf{var}[v_i|f]$  be the variance conditioned on  $f$ . A short calculation gives that

$$\begin{aligned} \mu_i(f) &= \mathbf{E}[v_i|f] = X_i [\gamma_i - p_i(f)(\gamma_i + \delta_i)], \\ \sigma_i^2(f) &= \mathbf{var}[v_i|f] = X_i^2 p_i(f)(1 - p_i(f))(\gamma_i + \delta_i)^2. \end{aligned}$$

Let  $v = \sum_i v_i$  be the value of the portfolio. Then, conditioned on  $f$ , we can compute its expectation  $\mu(f) = \mathbf{E}[v|f]$  as

$$\mu(f) = \mathbf{E}[v|f] = \sum_i \mu_i(f) = \sum_i X_i [\gamma_i - p_i(f)(\gamma_i + \delta_i)].$$

Since, conditioned on  $f$ ,  $v_i$  are independent, we can compute the variance  $\sigma^2(f) = \mathbf{var}[v|f]$ ,

$$\sigma^2(f) = \mathbf{var}[v|f] = \sum_i \sigma_i^2(f) = \sum_i X_i^2 p_i(f)(1 - p_i(f))(\gamma_i + \delta_i)^2.$$

Further, conditioned on  $f$ , since the  $v_i$  are *independent*, we can invoke the strong law of large numbers which gives that as  $n \rightarrow \infty$ , the distribution function of  $v$  converges a normal distribution function, i.e.

$$\mathbf{Pr}[v \leq V|f] \rightarrow \phi \left( \frac{V - \mu(f)}{\sigma(f)} \right),$$

where  $\phi(x)$  is the standard normal distribution function,  $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x ds e^{-\frac{1}{2}s^2}$ . There is a mild technical restriction which is that the strong law of large numbers

only will apply when the portfolio is not too concentrated. We will not dwell on such technicalities. Let  $G$  be the distribution function for the value  $v$  of the portfolio.  $G$  is now obtained by integrating with respect to  $f$ ,

$$G(v) = \int_{-\infty}^{\infty} dF(f) \phi\left(\frac{v - \mu(f)}{\sigma(f)}\right),$$

where

$$\begin{aligned} \mu(f) &= \sum_i X_i [\gamma_i - p_i(f)(\gamma_i + \delta_i)], \\ \sigma^2(f) &= \sum_i X_i^2 p_i(f)(1 - p_i(f))(\gamma_i + \delta_i)^2, \\ p_i(f) &= F_i\left(\frac{t_i - \lambda_i f}{\sqrt{1 - \lambda_i^2}}\right). \end{aligned}$$

Probably the most intuitive case to consider is the Gaussian copula, in which case  $f, \epsilon_i \sim N(0, 1)$ , i.e.  $F, F_i = \phi$ . In this case, we have

$$G(v) = \int_{-\infty}^{\infty} \frac{df}{\sqrt{2\pi}} e^{-\frac{1}{2}f^2} \phi\left(\frac{v - \mu(f)}{\sigma(f)}\right), \quad (3.1)$$

where

$$\begin{aligned} \mu(f) &= \sum_i X_i [\gamma_i - p_i(f)(\gamma_i + \delta_i)], \\ \sigma^2(f) &= \sum_i X_i^2 p_i(f)(1 - p_i(f))(\gamma_i + \delta_i)^2, \\ p_i(f) &= \phi\left(\frac{t_i - \lambda_i f}{\sqrt{1 - \lambda_i^2}}\right). \end{aligned}$$

The integral in (3.1) is very well behaved because  $\phi \in [0, 1]$  and so integrating against a Gaussian kernel is very easy to numerically approximate to more digits of precision than are financially relevant. Note that if there are more than one factor, then the integral in (3.1) becomes a multiple integral, and the logic still holds, though harder to work with. For a small number of factors one could use adaptive quadrature, to evaluate the integral, but when we have more than 3 dimensions, Monte-Carlo

delivers better computational performance.

### 3.3.2 Examples

Let's consider a simple yet realistic scenario, in which all deals have the same LGD *factor*,  $\delta_i = \delta$ , and the deals are constructed in the following simple way. In each deal, we try to make the same profit spread  $\gamma_i = \gamma$ . Further, each deal is constructed to have a particular unconditional expected value. To be concrete, let's set this unconditional expected value to 0, so  $\mathbf{E}[v_i] = 0$  (in general, one might desire some expected percentage profit,  $\mathbf{E}[v_i] = rX_i$ ), however the analysis will not be affected. Let  $p_i$  be the unconditional probability of default, then since  $d_i$  has an unconditional  $N(0, 1)$  distribution, we have the  $p_i = \phi(t_i)$ . Since  $\mathbf{E}[v_i] = rX_i = X_i(\gamma - p_i(\gamma + \delta)) = 0$ , we have that

$$p_i = \frac{\gamma - r}{\gamma + \delta}.$$

Thus, under these conditions, which are reasonably realistic,  $p_i = p$  is also independent of  $i$ , and since  $p_i = \phi(t_i)$ ,  $t_i = t$  is also independent of  $i$ . The interpretation of this is that to first approximation, there is some general background probability of default, which may be viewed as the market probability of default, which drives defaults across all sectors, companies, etc.

Further, we can also imagine in this one factor scenario, that there is some general market driving factor  $f$  which is driving all deals, with the same magnitude, i.e.,  $\lambda_i = \lambda$  is constant across the deals. We make this approximation just for illustration, and the machinery easily adapts to non-constant  $\lambda_i$ . In this setup, many simplifications occur in the integral (3.1), and we get

$$G(v) = \int_{-\infty}^{\infty} \frac{df}{\sqrt{2\pi}} e^{-\frac{1}{2}f^2} \phi\left(\frac{v - \mu(f)}{\sigma(f)}\right), \quad (3.2)$$

where

$$\begin{aligned}\mu(f) &= [\gamma - p(f)(\gamma + \delta)] \sum_i X_i, \\ \sigma^2(f) &= p(f)(1 - p(f))(\gamma + \delta)^2 \sum_i X_i^2, \\ p(f) &= \phi\left(\frac{t - \lambda f}{\sqrt{1 - \lambda^2}}\right), \\ t &= \phi^{-1}\left(\frac{\gamma - r}{\gamma + \delta}\right).\end{aligned}$$

After some manipulation, we obtain

$$G(v) = \int_{-\infty}^{\infty} \frac{df}{\sqrt{2\pi}} e^{-\frac{1}{2}f^2} \phi\left(\frac{v - x + y\phi(\alpha - \beta f)}{\sqrt{z\phi(\alpha - \beta f)(1 - \phi(\alpha - \beta f))}}\right), \quad (3.3)$$

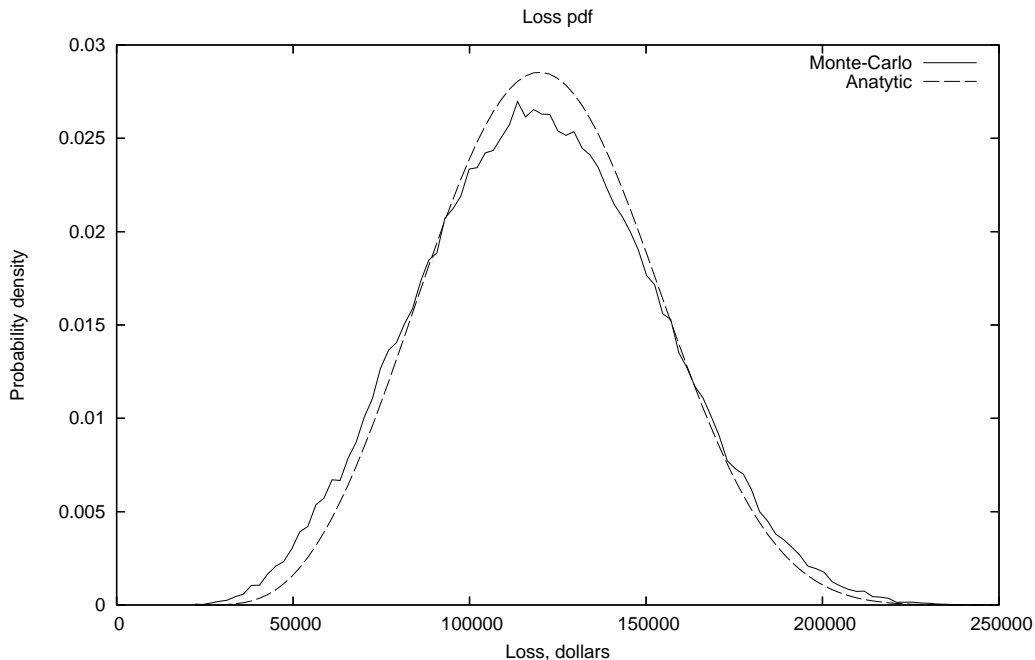
where  $x, y, z, \alpha, \beta$  are all constants given by

$$\begin{aligned}x &= \gamma \sum_i X_i, \\ y &= (\gamma + \delta) \sum_i X_i, \\ z &= (\gamma + \delta)^2 \sum_i X_i^2, \\ \alpha &= \frac{t}{\sqrt{1 - \lambda^2}}, \\ \beta &= \frac{\lambda}{\sqrt{1 - \lambda^2}},\end{aligned}$$

where  $t = \phi^{-1}\left(\frac{\gamma - r}{\gamma + \delta}\right)$ .

### 3.3.3 Simulation Experiment

We have run a simulation with a synthetic portfolio where the notional  $X_i = 500000$  for the  $n = 1000$  deals are generated randomly. The profit spread  $\gamma = 0.02$  (2% profit spread) and the LGD factor was set at 0.4 (40% LGD); market driven correlation factor was varied among the simulations ( $\lambda = 0.19$  is the choice of *MKMV*). All these parameters are reasonable based on current practice/assumptions. The results for the theory and Monte Carlo are virtually indistinguishable.



**Figure 3.1: Comparison of pdfs between Monte-Carlo and the analytic method**

For  $n = 1,000$ , the runtime for 1 million Monte Carlo simulations is about 45 minutes, whereas the theoretical calculation using the CLT takes about 5 seconds, almost independent of  $n$ . Thus the benefits of the theoretical calculation are clear in this simplified single period setting.

### 3.3.4 Adapting the Analysis to the Log-Normal Case

If instead, we have that the default variable is a “translated” log-normal,

$$d_i = e^{\lambda_i f + \sqrt{\chi^2 - \lambda_i^2} \epsilon_i} - e^{\frac{1}{2}\chi^2},$$

where  $\chi^2 = \log \frac{1+\sqrt{5}}{2}$ ; this choice of  $\chi^2$  is to ensure that  $\mathbf{E}[d_i] = 0$  and  $\mathbf{var}[d_i] = 1$ . Here, again,  $f$  is the driving factor for the market, and  $\epsilon_i$  is an idiosyncratic stochastic factor for  $D_i$ . The market  $S_m$  is also assumed to be log-normal, normalized to unit variance,  $S_m = e^{x^f}$ . A useful statistic for comparison with *MKMV* is the covariance,

$$\mathbf{cov}[d_i, S_m] = e^{x^2} (e^{x\lambda_i} - 1),$$

because *MKMV* defines

$$R = \frac{\mathbf{cov}[asset, market]}{\sqrt{\mathbf{var}[asset] \mathbf{var}[market]}},$$

which is one of the inputs to *MKMV*. Since  $\mathbf{var}[d_i] = \mathbf{var}[S_i] = 1$ , we have that

$$R = e^{\chi^2} (e^{\chi \lambda_i} - 1),$$

This equation allows us to convert the *MKMV*  $R$  to our  $\lambda$  notation ,or inverting,

$$\lambda_i = \frac{1}{\chi} \log(R e^{-\chi^2} + 1).$$

Once again, default occurs if  $d_i \leq t_i$ , iff  $d_i + e^{\frac{1}{2}\chi^2} \leq t_i + e^{\frac{1}{2}\chi^2}$ , iff  $\log(d_i + e^{\frac{1}{2}\chi^2}) \leq \log(t_i + e^{\frac{1}{2}\chi^2})$ . Since  $\log(t_i + e^{\frac{1}{2}\chi^2}) \sim N(0, \chi^2)$ , we have for the probability of default,

$$pd_i = \phi \left( \frac{\log(t_i + e^{\frac{1}{2}\chi^2})}{\chi} \right).$$

Inverting, we get

$$t_i = e^{\chi \phi^{-1}(pd_i)} - e^{\frac{1}{2}\chi^2}.$$

Note that

$$\log(d_i + e^{\frac{1}{2}\chi^2}) = \lambda_i f + \sqrt{\chi^2 - \lambda_i^2} \epsilon_i.$$

Conditioning on  $f$ , default occurs iff

$$\lambda_i f + \sqrt{\chi^2 - \lambda_i^2} \leq \log(t_i + e^{\frac{1}{2}\chi^2}),$$

and so

$$p_i(f) = \phi \left( \frac{\log(t_i + e^{\frac{1}{2}\chi^2}) - \lambda_i f}{\sqrt{\chi^2 - \lambda_i^2}} \right).$$

Since  $f$  is still a  $N(0,1)$  random variable, the remainder of the derivation carries through, and we have

$$G(v) = \int_{-\infty}^{\infty} \frac{df}{\sqrt{2\pi}} e^{-\frac{1}{2}f^2} \phi\left(\frac{v - \mu(f)}{\sigma(f)}\right), \quad (3.4)$$

where

$$\begin{aligned} \mu(f) &= \sum_i X_i [\gamma_i - p_i(f)(\gamma_i + \delta_i)], \\ \sigma^2(f) &= \sum_i X_i^2 p_i(f)(1 - p_i(f))(\gamma_i + \delta_i)^2, \\ p_i(f) &= \phi\left(\frac{\log(t_i + e^{\frac{1}{2}\chi^2}) - \lambda_i f}{\sqrt{\chi^2 - \lambda_i^2}}\right). \end{aligned}$$

In terms of  $pd_i$ , we can write  $p_i(f) = \left(\frac{\chi\phi^{-1}(pd_i) - \lambda_i f}{\sqrt{\chi^2 - \lambda_i^2}}\right)$ .

### 3.4 Continuous Time With Interest Rate Term Structure

So far, the correlated case has focused on the single period. In the multi-period case, which in the limit is the continuous setup, we need to consider the complete series of cashflows, and we will need to take into account the interest rate term structure  $P(t)$ , which we assume is given. It is useful to define  $R(t) = \int_0^t ds P(s)$ , and  $Q_i(t) = \int_0^t ds C_i(s)P(s)$ . All value/loss distributions will be in terms of present value dollars. Future values will be computed according to the assumed risk-neutral default probabilities and expectation of discounted cashflows.

#### 3.4.1 Notation

We summarize some notation here for convenience.

1.  $\tau$ , the time horizon for calculating the value distribution (typically  $\tau = 1$  year).
2.  $X_i$ : amount of outstanding capital at time 0.
3.  $T_i$ : maturity of deal  $i$ .

4.  $P(t)$ : the interest rate term structure (discount factor). For constant interest rate  $r$ ,  $P(t) = e^{-rt}$ . In general, the instantaneous spot rate  $r(t) = -\frac{d}{dt} \log P(t)$ .
5.  $R(t)$ :  $\int_0^t dsP(s)$ . For constant interest rate,  $R(t) = \frac{1}{r}(1 - e^{-rt})$ .
6.  $c_i$ : constant cashflow rate, i.e. for the uniformly amortized loan  $C_i(t) = \begin{cases} c_i & 0 \leq t \leq T_i, \\ 0 & \text{otherwise,} \end{cases}$ .
7.  $\gamma_i$ : spread fraction defined as  $\gamma_i = \frac{Q_i(T_i)}{X_i} - 1$ , which is the percentage profit from this deal assuming no default. It seems reasonable that  $\gamma_i \geq 0$ , though this is not a necessity.
8.  $\delta_i$ : the LGD parameter, where  $1 - \delta_i$  is the recovery rate of outstanding principal, where the outstanding principle at time  $t$ , denoted  $X_i(t)$  is given by  $X_i(t) = X_i(1 - \frac{Q_i(t)}{Q_i(T_i)})$ .
9.  $D_i$ : default indicator variable (as before).
10.  $d_i$ : default variable which drives  $D_i$ . We will make the multi-factor assumption  $d_i = \Omega(\sum_{k=1}^K \lambda_{ik} f_k + \beta_i \epsilon_i)$ , where the  $f$ 's and  $\epsilon$ 's are all independent, and  $\Omega$  is some invertible function. For the normally distributed case,  $\Omega(x) = x$  and the  $f$ 's and  $\epsilon$ 's are all  $\sim N(0, 1)$ . For the lognormal case,  $\Omega$  is an exponential function,  $\Omega(x) = e^x - e^{\frac{1}{2}\chi^2}$  where  $\chi^2 = \log \frac{1+\sqrt{5}}{2}$  is the log of the golden ratio. Let  $\{F_j\}, F_{\epsilon_i}, F_{d_i}$  be the distribution functions for  $\{f_j\}, \epsilon_i, d_i$ , and we will assume that  $\mathbf{E}[f_j] = \mathbf{E}[\epsilon_i] = 0$  and that  $\mathbf{E}[f_j^2] = \mathbf{E}[\epsilon_i^2] = 1$ . Depending on choices for these distributions (Gaussian, LogNormal, Student- $t$ , etc.) these distribution functions and their inverses may or may not be in closed form.
11.  $pd_i$  is the  $\tau$ -period unconditional probability of default. The survival rate  $\alpha_i$  defines the survival probability  $s_i(t) = e^{-\alpha_i t}$ , from which we identify  $\alpha_i \tau = -\log(1 - pd_i)$ . Remember that the default density is  $f_{D_i}(t) = \alpha_i e^{-\alpha_i t}$ .

**Assumption 1** *The default events  $\{D_i\}$  are dependent through the factors  $\{f_j\}$ . However, conditioned on default (in the interval  $[0, \tau]$ ), if the transition is to the*

default state ( $D_i = 1$ ), the time of default for each deal are independent random variables.

This assumption allows us to not have to worry about dependent stochastic processes for the driving variables, but rather only dependent state transitions. Note that this *is* an assumption which is made primarily to make the analytics tractable. However, it will probably have negligible practical impact. First we will elaborate on the model.

### 3.4.2 Cash Flows

The present value of all cashflows assuming no default is given by  $\int_0^\infty ds C_i(s)P(s)$ , and so we define

$$\gamma_i = \frac{1}{X_i} \int_0^\infty ds C_i(s)P(s) - 1 = \frac{Q_i(\infty)}{X_i} - 1.$$

For the simple uniformly amortized loan,

$$C_i(t) = \begin{cases} c_i & 0 \leq t \leq T_i, \\ 0 & \text{otherwise,} \end{cases}$$

in which case  $\gamma_i = \frac{c_i R(T_i)}{X_i} - 1$ . For the uniformly amortized bullet loan,

$$C_i(t) = \begin{cases} c_i + B_i \delta(t - T_i) & 0 \leq t \leq T_i^+, \\ 0 & \text{otherwise,} \end{cases}$$

in which case  $\gamma_i = \frac{1}{X_i}(c_i R(T_i) + B_i P(T_i)) - 1$ . It seems reasonable that  $\gamma_i$  should be positive, which will naturally be the case at inception of the loan, and will therefore continue to be the case when there is no interest rate risk.

Lets now define the outstanding capital at time  $t$ ,  $X_i(t)$ . We write  $C_i(t) = cap_i(t) + int_i(t)$ , where  $cap_i(t)$  is the rate at which capital is being paid, and  $int_i(t)$  is the rate at which interest is being paid. In principle  $cap_i(t)$  and  $int_i(t)$  can be specified arbitrarily, i.e. as long as  $\int_0^\infty cap_i(t) = X_i$ , i.e., eventually the entire

outstanding capital gets paid. The natural choice for  $cap_i(t)$  for the bullet loan is  $cap_i(t) = 0$  except for the one last bullet payment which pays all the capital, i.e.,  $B_i = X_i$ , i.e.,  $C_i(t) = X_i\delta(t - T_i)$ . For the uniformly amortized loan, we may use the following argument to define  $cap_i(t)$ . During the interval  $(t, dt)$  payment  $C_i(t)dt$  has present value  $P(t)C_i(t)dt$ , and we pay off capital  $cap_i(t)dt$ . Hence, the spread is  $P(t)C_i(t)dt - cap_i(t)dt$ . We set this to  $\gamma_i cap_i(t)dt$  to enforce the fact that we uniformly collect our spread over the duration of the loan. Hence,

$$cap_i(t) = \frac{C_i(t)P(t)}{1 + \gamma_i} = \frac{X_i C_i(t)P(t)}{Q_i(\infty)},$$

and  $int_i(t) = C_i(t) - cap_i(t)$ . We now use  $X_i(t) = X_i - \int_0^t ds cap_i(s)$  to obtain the outstanding capital at time  $t$ ,

$$X_i(t) = X_i \cdot \left(1 - \frac{Q_i(t)}{Q_i(\infty)}\right).$$

The outstanding capital will be needed to compute the loss given default based on the recovery rate.

### 3.4.3 A Single Deal

We notice that conditioned on  $\mathbf{f}$ , the analysis for the correlated case in the limit of large  $n$  which invokes the Central Limit Theorem only requires knowledge of the mean value  $\mu_i(\mathbf{f})$  and the variance,  $\sigma_i^2(\mathbf{f})$ , for a single deal. To proceed, let's restate the definition of the driving variable  $d_i$ ,

$$d_i = \Omega(\boldsymbol{\lambda}_i^t \mathbf{f} + \beta_i \epsilon_i).$$

Default occurs if  $d_i \leq t_i$ , which occurs if and only if

$$\epsilon_i \leq \frac{\Omega^{-1}(t_i) - \boldsymbol{\lambda}_i^t \mathbf{f}}{\beta_i}.$$

Thus,

$$\Pr [d_i \leq t_i | \mathbf{f}] = \Pr \left[ \epsilon_i \leq \frac{\Omega^{-1}(t_i) - \boldsymbol{\lambda}_i^t \mathbf{f}}{\beta_i} \right] = F_{\epsilon_i} \left( \frac{\Omega^{-1}(t_i) - \boldsymbol{\lambda}_i^t \mathbf{f}}{\beta_i} \right).$$

We have a relationship between  $pd_i$  and  $t_i$ , since  $pd_i = \mathbf{Pr} [d_i \leq t_i] = F_{d_i}(t_i)$ ,

$$t_i = F_{d_i}^{-1}(pd_i).$$

Our basic assumption is that conditioned on the factor  $\mathbf{f}$ , the deal defaults are independent. The factor  $\mathbf{f}$  serves to specify the probability of default,  $pd_i(\mathbf{f})$ ,

$$\begin{aligned} pd_i(\mathbf{f}) &= F_{\epsilon_i} \left( \frac{\Omega^{-1}(t_i) - \lambda_i^t \mathbf{f}}{\beta_i} \right), \\ &= F_{\epsilon_i} \left( \frac{\Omega^{-1}(F_{d_i}^{-1}(pd_i)) - \lambda_i^t \mathbf{f}}{\beta_i} \right). \end{aligned}$$

From  $pd_i(\mathbf{f})$ , we get

$$\alpha_i(\mathbf{f}) = \frac{-\log(1 - pd_i(\mathbf{f}))}{\tau}.$$

Thus, in our model, if we are to be precise, it is not the defaults which are correlated, but the *default rates*  $\alpha_i(\mathbf{f})$  which are correlated.

Based on our previous discussion we will make use of the following notion of outstanding capital at time  $t$ , which is natural and can apply equally well to both the bullet loan and the uniformly amortized loan to within reasonable approximation.

**Assumption 2** *The outstanding capital  $X_i(t)$  is that fraction of the initial outstanding capital  $X_i$  equal to the cash flows (in present value) which remain to be paid divided by the total cashflows (in present value),*

$$X_i(t) = X_i \frac{\int_t^\infty ds C_i(s)P(s)}{\int_0^\infty ds C_i(s)P(s)} = X_i \left( 1 - \frac{Q_i(t)}{Q_i(\infty)} \right) = \frac{\int_t^\infty ds C_i(s)P(s)}{1 + \gamma_i}.$$

*In present value dollars,  $(1 - \delta_i)X_i(t)$  of the outstanding capital is recovered when the loan defaults at time  $t$ .*

**Theorem 3.4.1** *In present value dollars, we have for the first two central moments*

of the loss distribution, conditioned on  $\mathbf{f}$ ,

$$\mathbf{E}[L_i(\tau)|\mathbf{f}] = \delta'_i(Q_i(\infty, 0) - Q_i(\infty, \alpha_i)) - \gamma_i X_i, \quad (3.5)$$

$$\begin{aligned} \frac{\mathbf{var}[L_i(\tau|\mathbf{f})]}{\delta_i'^2} &= (e^{\alpha_i \tau} - 1)(Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i))^2 + \\ &+ 2Q_i(\tau, 0)(Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i)) + G_i(\tau, \alpha_i) - Q_i^2(\infty, \alpha_i), \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} Q_i(t, \alpha) &= \int_0^t ds C_i(s)P(s)e^{-\alpha s}, \\ G_i(\tau, \alpha) &= \int_0^\tau ds \int_0^\tau dz C_i(s)P(s)C_i(z)P(z)e^{-\alpha_i \max(s,z)}, \\ pd_i(\mathbf{f}) &= F_{\epsilon_i} \left( \frac{\Omega^{-1}(F_{d_i}^{-1}(pd_i)) - \boldsymbol{\lambda}_i^t \mathbf{f}}{\beta_i} \right), \\ \alpha_i = \alpha_i(\mathbf{f}) &= \frac{-\log(1 - pd_i(\mathbf{f}))}{\tau}, \\ \delta'_i &= \frac{\delta_i + \gamma_i}{1 + \gamma_i}, \\ \gamma_i &= \frac{Q_i(\infty, 0)}{X_i} - 1. \end{aligned}$$

**The Gaussian Case:**  $\Omega(x) = x$ ;  $F_{d_i}(x) = F_{\epsilon_i}(x) = \phi(x)$ ;  $\beta_i = \sqrt{1 - \|\boldsymbol{\lambda}_i\|^2}$ , and so

$$pd_i(\mathbf{f}) = \phi \left( \frac{\phi^{-1}(pd_i) - \boldsymbol{\lambda}_i^t \mathbf{f}}{\sqrt{1 - \|\boldsymbol{\lambda}_i\|^2}} \right).$$

**The Log-Normal Case:**  $\Omega(x) = e^x - e^{\frac{1}{2}\chi^2}$ ;  $F_{d_i}(x) = \phi\left(\frac{1}{\chi} \log(x + e^{\frac{1}{2}\chi^2})\right)$  and  $F_{\epsilon_i}(x) = \phi(x)$ ;  $\beta_i = \sqrt{\chi^2 - \|\boldsymbol{\lambda}_i\|^2}$ , and so

$$pd_i(\mathbf{f}) = \phi \left( \frac{\chi \phi^{-1}(pd_i) - \boldsymbol{\lambda}_i^t \mathbf{f}}{\sqrt{\chi^2 - \|\boldsymbol{\lambda}_i\|^2}} \right),$$

where  $\chi^2 = \log \frac{1+\sqrt{5}}{2}$ . The proof is given in the appendix

### 3.4.4 Special case of constant interest rate

For a constant interest rate:  $P(t) = e^{-rt}$ . We will consider the uniform bullet loan, so  $C_i(t) = c_i$  except for the bullet payment at time  $T_i$ . We just need to compute  $Q_i(t, \alpha)$  and  $G_i(t, \alpha)$  so that  $\mathbf{E}[L_i(\tau)|\mathbf{f}]$  and  $\mathbf{var}[L_i(\tau)|\mathbf{f}]$  can be computed.

**Uniform Bullet Loan.** We will use the notation  $[[bool]]$  for a boolean expression  $bool$  to be 1 if  $bool$  is TRUE and 0 if  $bool$  is FALSE. The only difference between the uniformly amortized loan and the uniform bullet loan is a single bullet payment  $B_i$  at  $T_i$ . Typically,  $B_i = X_i$ . Thus,

$$\begin{aligned} Q_i^B(t, \alpha) &= \int_0^t ds C_i(s)P(s)e^{-\alpha s}, \\ &= \int_0^{\min(T_i^+, t)} ds (c_i + B_i\delta(s - T_i))e^{-(\alpha+r)s}, \\ &= \frac{c_i}{\alpha + r} (1 - e^{-(\alpha+r)\min(T_i, t)}) + B_i e^{-(\alpha+r)T_i} [[T_i \leq t]], \end{aligned}$$

$$\begin{aligned} G_i^B(t, \alpha) &= \int_0^\tau ds \int_0^\tau dz C_i(s)P(s)C_i(z)P(z)e^{-\alpha_i \max(s, z)}, \\ &= \int_0^{\min(\tau, T_i^+)} ds (c_i + B_i\delta(s - T_i))e^{-rs} \cdot \\ &\quad \cdot \int_0^{\min(\tau, T_i^+)} dz (c_i + B_i\delta(z - T_i))e^{-rz} e^{-\alpha_i \max(s, z)}, \\ &= 2c_i^2 \int_0^{\min(\tau, T_i^+)} ds \int_0^s dz e^{-(r+\alpha_i)s} e^{-rz} + 2B_i c_i e^{-(r+\alpha_i)T_i} [[T_i \leq \tau]] \int_0^{T_i} ds e^{-rs} \\ &\quad + B_i^2 e^{-(2r+\alpha_i)T_i} [[T_i \leq \tau]], \\ &= \frac{2c_i^2}{r} \left( \frac{1 - e^{-(r+\alpha_i)\min(\tau, T_i)}}{r + \alpha_i} - \frac{1 - e^{-(2r+\alpha_i)\min(\tau, T_i)}}{2r + \alpha_i} \right) \\ &\quad + B_i e^{-(r+\alpha_i)T_i} \left( \frac{2c_i}{r} (1 - e^{-rT_i}) + B_i e^{-rT_i} \right) [[T_i \leq \tau]]. \end{aligned}$$

The result for the uniformly amortized loan is obtained by setting  $B_i = 0$ .

### 3.4.5 Multiple Correlated Deals with Term Structure

We now proceed exactly as in the case of the single period.

$$L(\tau) = \sum_i L_i(\tau),$$

from which it is immediate that  $L(\tau|\mathbf{f}) = \sum_i L_i(\tau|\mathbf{f})$ . Therefore,

$$\mu(\mathbf{f}) = \mathbf{E}[L(\tau|\mathbf{f})] = \sum_i \mathbf{E}[L_i(\tau|\mathbf{f})] = \sum_i \mu_i(\mathbf{f}).$$

Since  $L_i(\tau|\mathbf{f})$  and  $L_j(\tau|\mathbf{f})$  are independent random variables for  $i \neq j$ , we immediately have that

$$\sigma^2(\mathbf{f}) = \mathbf{var}[L(\tau|\mathbf{f})] = \sum_i \mathbf{var}[L_i(\tau|\mathbf{f})] = \sum_i \sigma_i^2(\mathbf{f}).$$

Notice that  $\mu_i(\mathbf{f}), \sigma_i^2(\mathbf{f})$  are functions of  $f$  only through  $\lambda_i \cdot \mathbf{f}$ . Exactly analogous to the single period case, we now have that

$$G_L(h) = P(L \leq h) = \int_{-\infty}^{\infty} d\mathbf{f} \frac{1}{(2\pi)^{K/2}} e^{-\frac{1}{2}f^2} \phi\left(\frac{h - \mu(\mathbf{f})}{\sigma(\mathbf{f})}\right),$$

which is a  $K$ -dimensional very well behaved integral, where  $K$  is the number of factors. In the one factor case, things are numerically simple though. The issues come (numerically) when we are interested in (say) 100 factors. In such a case, this integral would have to be done using a technique like Monte-Carlo.

### 3.4.6 Correlated Defaults (not Default Rates)

We now consider instead of correlated default rates, correlated defaults. The only modification is that in the case of no default, the default rate of each deal then reverts to the normal default rate  $pd_i$ , unconditioned on  $\mathbf{f}$ . This only affects  $L(\tau|\text{no default})$ .

$$\begin{aligned} L_i(\tau|\text{no default}) &= \delta'_i \int_{\tau}^{\infty} dz C_i(z) P(z) (1 - e^{-\alpha_i(z-\tau)}) - \gamma_i X_i, \\ L_i(\tau|\text{default in } (t, dt)) &= \delta'_i (Q_i(\infty) - Q_i(t)) - \gamma_i X_i, \end{aligned}$$

where  $\alpha_i = -\log(1 - pd_i)/\tau$ . In the previous computations, only  $\alpha_i(\mathbf{f}) = -\log(1 - pd_i(\mathbf{f}))/\tau$  appeared in the calculations. In this discussion, we will have to distinguish between  $\alpha_i$  and  $\alpha_i(\mathbf{f})$ . Following the same process as the previous section, we arrive without much difficulty at the following expression for  $\mathbf{E}[L_i|\mathbf{f}]$ ,

$$\frac{\mathbf{E}[L_i|\mathbf{f}]}{\delta'_i} = Q_i(\infty, 0) - e^{-(\alpha_i(\mathbf{f}) - \alpha_i)\tau} Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i(\mathbf{f})) + e^{-(\alpha_i(\mathbf{f}) - \alpha_i)\tau} Q_i(\tau, \alpha_i) - \frac{\gamma_i X_i}{\delta'_i}.$$

When  $\alpha_i(\mathbf{f}) = \alpha_i$ , i.e. the default rate before  $\tau$  (correlated default) equals the default rate after  $\tau$ , we recover the previous result. The main difference between the two analyses is that the correlated default process “restarts” after time  $\tau$  in the event of no default. Setting  $Y_i = L_i + \gamma_i X_i - \delta'_i Q_i(\infty, 0)$  as before, we compute  $\mathbf{var}[L_i] = \mathbf{var}[Y_i]$  as,

$$\begin{aligned} \frac{\mathbf{var}[L_i|\mathbf{f}]}{\delta_i'^2} &= G(\tau, \alpha_i(\mathbf{f})) + 2e^{-(\alpha_i(\mathbf{f}) - \alpha_i)\tau} (Q_i(\tau, 0) - Q_i(\tau, \alpha_i(\mathbf{f}))) (Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i)) \\ &\quad - Q_i^2(\tau, \alpha_i(\mathbf{f})) + e^{-2(\alpha_i(\mathbf{f}) - \alpha_i)\tau} (e^{\alpha_i(\mathbf{f})\tau} - 1) (Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i))^2. \end{aligned}$$

### 3.4.6.1 Expected Loss (EL)

We can easily compute the expected loss. From (3.4), we have that

$$\mathbf{E}[L_i] = \delta'_i (Q_i(\infty, 0) - Q_i(\infty, \alpha_i)) - \gamma_i X_i,$$

where  $\alpha_i = -\frac{\log(1 - pd_i)}{\tau}$ . We thus have that

$$\mathbf{E}[L] = \sum_i \mathbf{E}[L_i] = \sum_i \delta'_i (Q_i(\infty, 0) - Q_i(\infty, \alpha_i)) - \gamma_i X_i.$$

## CHAPTER 4

### Combined Credit and Interest Rate Risk

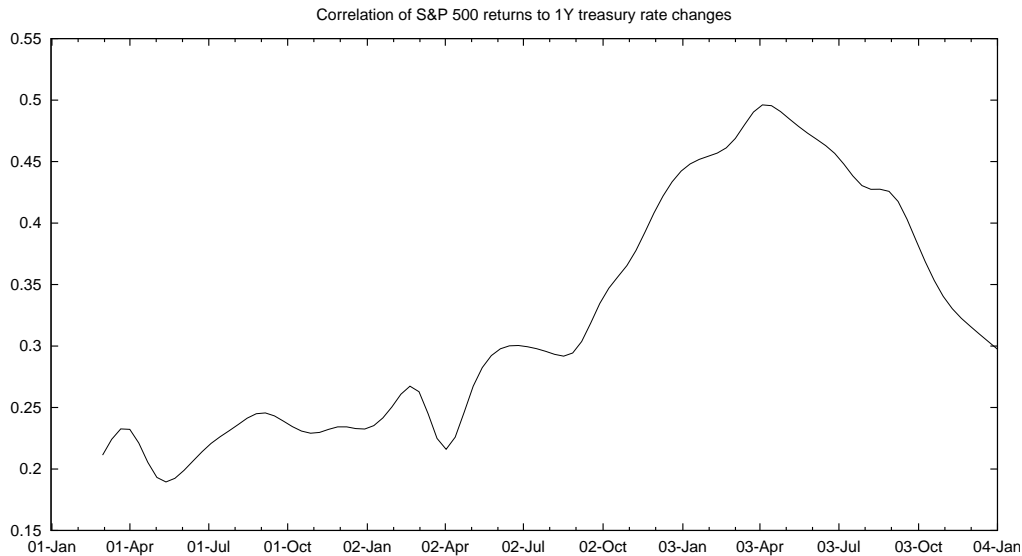
Usually lenders use separate models for credit and interest rate risks. However, in the real world, credit risk and interest rate risks are not independent. For example during recessions the number of defaults goes up and interest rates usually go down. Also sharp interest rate increases could affect the ability of borrowers to repay variable rate loans. The dependence between the interest rate and loans' default rates makes estimation of credit portfolio losses harder. Banks and other lenders use various ad-hoc methods to combine credit risk and interest rate risk for total loss estimation. In many cases those ad-hoc methods don't have any theoretical justification, and have less than desirable accuracy. We propose an integrated model for credit and interest rate risks, which addresses the issues described above. We studied the dependence between S&P 500 index – an important market factor and US treasury interest rates and found out that they are correlated. Correlations between S&P 500 daily returns and 1 Year Treasury rate returns are in range of  $[-0.51, 0.54]$  over the last 40 years (from 1967 - 2007) and the correlation changes slowly over time. We notice, for example, that during recessions the stock market goes down (and the number of defaults goes up) and interest rates usually go down. So we see that the assumption of non-zero correlation between stock market and interest rate is a valid one, and we need to include it into our model.

#### 4.1 A Stochastic Process for Interest Rates and Defaults

We model the dependence of the interest rate and the number of defaults by “tying” the interest rate to the market factor.

$$r(t) = r_0 e^{\alpha_r f(t) + \mu_r t + \sigma_r W_r(t)},$$

$\mu_r$  and  $\sigma_r$  are the mean and variance of the interest rate;  $\alpha_r$  specifies the dependence between the market factor and the interest rate;  $W_r(t)$  is a standard Brownian motion. The market factor has the following formula:



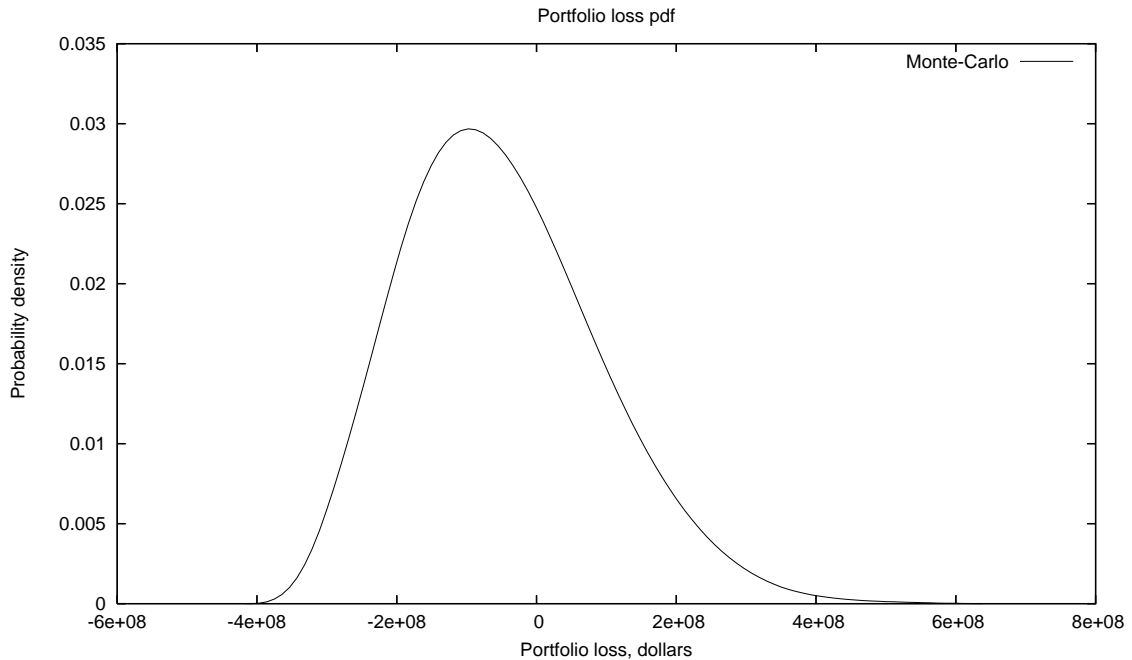
**Figure 4.1: Correlation between 1Y treasury interest rate changes and changes for S&P 500 index using 1 year lookback window**

$$f(t) = f_0 + \mu_f t + \sigma_f W_f(t),$$

Where  $\mu_f$  and  $\sigma_f$  are the mean and variance of the market factor;  $W_f(t)$  is a standard Brownian motion, independent of  $W_r(t)$ . The default variable for loan  $i$  has the following dynamics:

$$d_i(t) = \lambda_i f(t) + \sigma_{d_i} W_{d_i}(t).$$

The correlation  $\lambda_i$  specifies dependence between the market factor and the default variable,  $\sigma_{d_i}$  - is the coefficient which makes the default's variable total variance  $\sigma^2(d_i(t)) = t$ ,  $W_{d_i}(t)$  is a standard Brownian motion. As in the basic model, default happens when  $d_i(t) < t_i$ , where  $t_i$  is a threshold which has a negative value. The threshold  $t_i$  is calculated in such a way as to make the  $Pr(d_i(t) < t_i) = pd_i$  for a market factor which has the standard drift and 0 variance. In our current model, the market factor and the interest rate are separate variables, and the interest rate may depend on the market factor, so it would appear that high (low) values of the interest rate correspond to high (low) values of the market factor. Since the default variable also depends on the market factor there is an indirect dependence of the



**Figure 4.2: Probability density of portfolio losses**

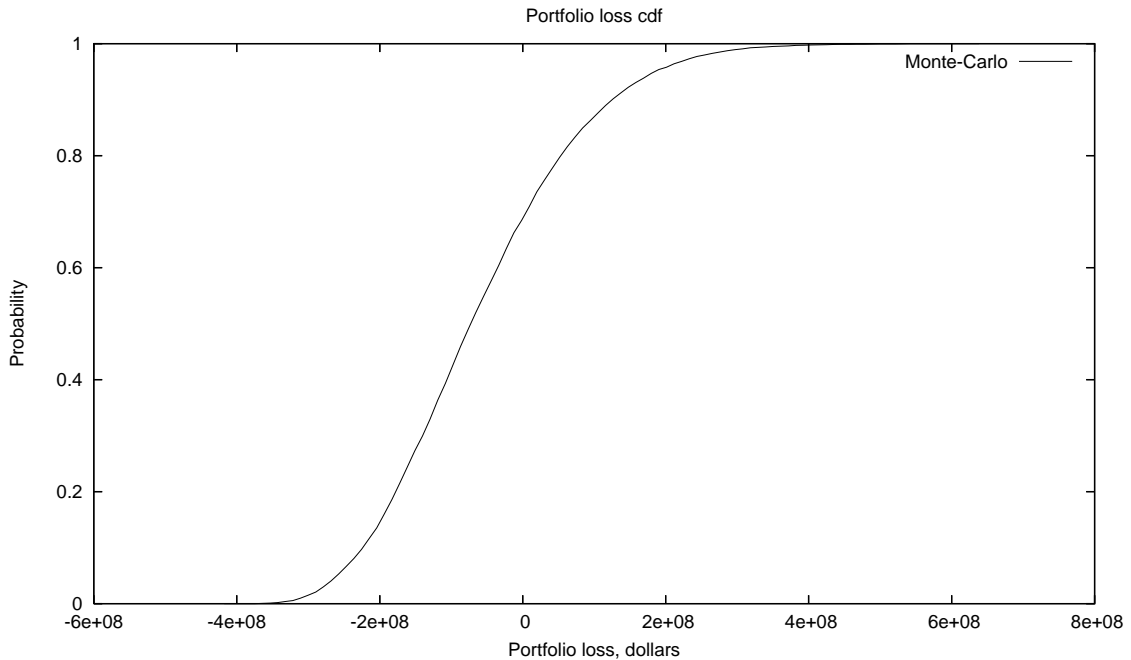
**Table 4.1: Values-at-risk for different economic scenarios, values for 99% VaR**

Economic scenario	Interest rate risk	Credit risk	Combined risk
Normal scenario	-2.649 B	0.366 B	0.563 B
Low interest rate volatility	-2.849 B	0.366 B	0.43 B
Market drifting down	-2.724 B	2.873 B	3.443 B

default event on the interest rate. However, conditioned on the market factor path, the interest rate and default variables are independent. This will be an important observation in the semi-analytic approach.

## 4.2 Monte-Carlo simulation

Our first algorithm to compute the loss distribution will be via Monte-Carlo. The first step to using Monte-Carlo is to generate a Monte-Carlo path, i.e. a realization for all the random quantities. In our model we have a random path for the factor  $f$ , which is the driver. Given the path for  $f(t)$  we can generate a path for the



**Figure 4.3: Cumulative probability density of portfolio losses**

interest rate  $r(t)$  and a path for each default variable  $d_i(t)$ . Let the size of the time step to be  $\delta t$ . The complete algorithm to generate a single Monte-Carlo realization is below.

We have two time horizons: credit risk time horizon (standard value is 1 year), and interest rate time horizon (standard value is 90 days). In the credit risk time horizon the default variables are simulated. In the interest rate time horizon the interest rate is simulated, after the end of the interest rate time horizon, the interest rate remains fixed.

We can simulate a random path which consists of values of market factors, default variables and the interest rate in time.

For each path we can exactly calculate the value of the portfolio. Generating many random paths (10 000+), we can obtain a histogram for loan portfolio's value, which will approximate the pdf of portfolio value. That is the main idea of the Monte-Carlo approach for the model.

We simulate a market factor and default variables for 365 days. We recalculate all the variables daily. Interest rate is recalculated daily from the beginning of the

simulation up until 90 days. After 90 days the interest rate remains fixed. If a default variable goes below its default threshold, then the loan is in default and we calculate the xrecovery rate of its remaining cashflows.

### 4.3 Separate estimation of the credit and interest rate risks

We can use the model to estimate the credit risk or the interest rate risk separately. If we set the drift ( $\mu_r$ ) and diffusion ( $\sigma_f$ ) coefficients of the interest rate to zero, then the interest rate will be constant and the interest rate risk will be eliminated, leaving only the credit risk. Alternatively if we want to estimate only the interest rate risk, we can set the default thresholds for the loans to large negative values (negative billions). This will virtually eliminate the possibility of a default and therefore will eliminate the credit risk, leaving the interest rate risk only.

#### 4.3.1 Brownian Bridge

We use a finite number of time points to evaluate default variables. It is possible that a continuous default variable goes below the threshold between consecutive time points and returns back above the threshold. We need to calculate a probability of the default variable moving below the threshold and then moving above the threshold between consecutive time points. We use the following formula for the Brownian Bridge breaching the threshold:

$$p_{BB} = e^{-\frac{2(-l+y_1)(-l+y_2)}{\sigma^2 \Delta t}}.$$

$y_1$  - the value of the Brownian motion at time point k with time  $t_k$

$y_2$  - the value of the Brownian motion at time point k+1 with time  $t_{k+1}$

$\Delta t = t_{k+1} - t_k$

$\sigma$  - standard deviation of the Brownian motion

$l$  - the value of the threshold

#### 4.3.2 Performance optimization

In order to improve the performance of the Monte-Carlo simulation we introduce variable time step

**Table 4.2: Performance for 10000 Monte-Carlo runs for different optimization methods for a 4-core Intel CPU 2.66 GHz**

Portfolio size, loans	No optimization	Variable time step	Variable time step + Multi-core
35000	35903 s	3043 s	762 s
10000	11114 s	954 s	236 s
5000	5397 s	471 s	117 s

For each loan we calculate an optimal time step, which depends on loan's probability of default

For example for a loan with high probability of default, the default variable will be recalculated and checked against the default threshold daily. For a very low-risk loan the default probability will be calculated once in 61 days. When a loan's default variable approaches its default threshold, its time step will become smaller

For real-world applications, the solution for the model should run reasonably fast. Loan portfolios consisting of hundreds of thousands of loans may take a long time for process for risk evaluation. We implemented several optimizations in order to improve the performance of the algorithms:

1. The production version of the code written in C++
2. Pre-allocation of memory for all containers, pre-calculation of commonly used values, etc
3. For Monte-Carlo approach we use variable time step for credit risk. When loan's probability of default is low then the new value of the default variable is calculated infrequently (once in 10 days), when pd is high - each day. This approach trades very little accuracy for a significant speedup

For both solution approaches computations are divided between multiple threads using a multithreading library. Multithreading achieves very significant speedup on computers with multiple CPUs or multiple cores. The number of threads in the software implementation is flexible and by default equals to the number of CPU cores in the system. The implementation of multi-threading has separate programming layers for multi-threading code and computational code and allows to modify the programs to work on clusters without changes in model code

**Table 4.3: Notation summary**

Notation	Description
$f(t)$	The market factor
$d_i(t)$	The default variable
$r(t)$	The interest rate
$\mu_f$	The drift of the marker factor
$\sigma_f$	The diffusion coefficient of the market factor
$\lambda_i$	The correlation between the default variable of the loan $i$ and the market factor
$\sigma_{d_i}$	The diffusion coefficient of the default variable of the loan $i$
$r_0$	The initial interest rate
$\mu_r$	The drift coefficient of the interest rate
$\beta$	The correlation between the interest rate and the market factor
$\sigma_r$	The diffusion coefficient of the interest rate
$c_i$	The cashflow rate for the loan $i$
$X_i$	The principal of the loan $i$
$B_i$	The bullet part of principal for the loan $i$
$\delta_i$	The loss given default rate for the loan $i$
$T_i$	The default threshold for the loan $i$
$\gamma_i$	The interest rate spread for the loan $i$

```

foreach loan i do
  |  $t_i = \text{GenerateDefaultThreshold}(i); \text{active}_i = \text{true};$ 
end
 $\text{cash} = 0;$ 
for  $t \leftarrow 1$  to DaysInCreditHorizon do
  |  $\text{GenerateValueOfTheMarketFactor } f(t);$ 
  if  $t \leq \text{DaysInInterestRateHorizon}$  then
    |  $\text{generate } r(t);$ 
  else
    |  $r(t) = r(t - 1);$ 
  end
  foreach loan i do
    | if  $\text{active}_i$  then
      | if LoanMatured(i) then
        |  $\text{active}_i = \text{false};$ 
        |  $\text{cash} = \text{cash} + B_i;$ 
      else
        |  $\text{GenerateTheDefaultVariabled}_i(t);$ 
        | if  $d_i \leq t_i$  then
          |  $\text{active}_i = \text{false};$ 
          |  $\text{cash} =$ 
          |  $\text{cash} + (1 - \text{lambda}_i) \cdot \text{DiscountedFutureCashflows}(i)$ 
        end
      end
      |  $\text{cash} = \text{cash} + c_i / \text{DaysInCreditHorizon};$ 
    end
  end
  |  $\text{cash} = \text{cash} * \exp(r(t) / \text{DaysInCreditHorizon})$ 
end
foreach loan i do
  | if  $\text{active}_i$  then
    |  $\text{cash} = \text{cash} + \text{ValueOfRemainingLoan}(i);$ 
  end
end
return cash;

```

Algorithm 1: Monte-Carlo pseudocode

## CHAPTER 5

### Loan's value after credit horizon

During the simulation described in the previous chapter we model the behavior of default variables during the credit time horizon. We collect cashflows from loans during the simulation. At the end of the simulation we may still have loans in the portfolio which didn't mature and didn't default during the simulation. Some parameters of those loans changed from their parameters in the initial portfolio: the times to maturity are decreased, the probabilities of default changed due to changes in the values of default variables (the default variables may now be closer to or further away from the default thresholds). In order to calculate the value of the portfolio at the end of the simulation time horizon, we need to calculate the values of the remaining loans. We assume that the value of a loan at the end of the time horizon is the sum of discounted cashflows starting at the end of the credit time horizon with the new probability of default taken into account. For simplicity we set the time at the end of the credit time horizon to zero.

$$Value(\text{Portfolio}) = Value(\text{Cash-after-simulation}) + Value(\text{Outstanding-loans}).$$

The sum of the loans's discounted cashflows up to time  $t$  can be expressed as:

$$\begin{aligned} Q_i(t) &= \int_0^t C_i(s)P_i(s)ds = \int_0^t c_iP_i(s)ds = c_i \int_0^t e^{-rs}ds[[t \leq T_i]] + \delta(t - T_i)B_i e^{-rt} \\ &= \left( c_i \frac{1 - e^{-rt}}{r} + \delta(t - T_i)B_i e^{-rT_i} \right) [[t \leq T_i]] \end{aligned}$$

$$Q_i(\infty) = Q_i(T_i) = c_i \frac{1 - e^{-rT_i}}{r} + B_i e^{-rT_i}.$$

In our framework we can divide any loan into two parts: a pure amortized part and a pure bullet part. In the case of a default at time  $t_d$  an obligor owes the unpaid amount of the amortized part and the bullet amount. We recover a fraction  $(1 - \delta_i)$  of the owed amount.

The amortized loan repayment can be divided into two parts: the first part is the amortized loan amount  $A_i = (X_i - B_i)$  which is placed in bank account 1 and it collects interest with rate  $r$ .

$$b_{1i}(t) = A_i e^{rt}.$$

The second part is the initially empty bank account which receives constant cashflows with rate  $c_A = c_i - B_i r$  and it grows with the interest rate  $r$ . This is the cash that is accruing after payment of interest on the bullet payment. At the maturity, both accounts should be equal.

$$b_{2i}(t) = c_A \frac{e^{rT_i} - 1}{r}.$$

The loan is paid off when the amounts on those two bank accounts are equal:

$$b_{1i}(t) = b_{2i}(t),$$

$$c_{Ai} \frac{e^{rT_i} - 1}{r} = A_i e^{rT_i},$$

$$c_{Ai} = \frac{A_i \cdot r \cdot e^{rT_i}}{e^{rT_i} - 1}.$$

If a default occurs before maturity at time  $t < T_i$ , then  $b_{1i}(t) > b_{2i}(t)$ , and the unpaid part of the amortized loan is:

$$S(t, A_i, r) = b_{1i}(t) - b_{2i}(t) = A_i e^{rt} - c_{Ai} \frac{e^{rt} - 1}{r}.$$

Using the expression for  $c_{Ai}$ , and after some algebra, we find that

$$S(t, X_i, B_i, r) = (X_i - B_i) \frac{1 - e^{-r(T_i - t)}}{1 - e^{-rT_i}}.$$

Any loan in our model can be represented as a combination of two simple loans:

1) pure amortized loan and 2) pure bullet loan. We assume that loan's cashflow  $c_i$

pays for interest on bullet first, then the interest on amortized part is paid and the remaining fraction of the cashflow pays the principal for the amortized loan. So we can express the remaining value of the loan as:  $X_i(t) = B_i e^{-r(T_i-t)} + S(t, X_i, B_i, r)$ . This assumption is realistic and widely used by banks and other lenders. In chapter 3 we introduced more general definition of outstanding capital which is defined as ratio of unpaid discounted cashflows to all discounted cashflows multiplied by loan's principal. We will use the definition for outstanding capital from this chapter, because it is widely used in practice.

For a loan we can have two cases: the case of default of the loan and the case of the loan paid off by the obligor. In the case of default at time  $t_d$  the loan has value:

$$V_{1i}(t_d) = Q_i(t_d) + (1 - \delta_i)(S(t, X_i, B_i, r) + B_i)e^{-rt_d}. \quad (5.1)$$

If the loan is paid off (i.e. if  $t_d > T_i$ ), the loan value is defined as:

$$V_{2i}(t_d) = Q_i(T_i).$$

Using the default time distribution, we note that the default time  $t_d$  should be lower than the loan maturity time  $T_i$ . Since we assume that the default distribution is the risk neutral default distribution, the value of the loan in the case of default is defined as:

$$\begin{aligned} V_{1i} &= \int_0^{T_i} \alpha_i(f) e^{-\alpha_i(f)t} V_{1i}(t) dt \\ &= \int_0^{T_i} \alpha_i(f) e^{-\alpha_i(f)t} \left[ Q_i(t_d) + (1 - \delta_i) \left( (X_i - B_i) \frac{1 - e^{-r(T-t_d)}}{1 - e^{-rT_i}} + B_i \right) e^{-rt_d} \right] dt. \end{aligned}$$

The value of the loan in the case of no default is defined as:

$$\begin{aligned} V_{2i} &= \int_{T_i}^{\infty} \alpha_i(f) e^{-\alpha_i(f)t} V_{2i}(t) dt, \\ V_{2i} &= \int_{T_i}^{\infty} \alpha_i(f) e^{-\alpha_i(f)t} \left( c_i \frac{1 - e^{-rT}}{r} + B_i \right) dt. \end{aligned}$$

Note that  $V_{2i}(t)$  is a constant, independent of  $t$ . The total value of the loan is given by the sum  $V_i = V_1 + V_2$  and so we obtain:

$$V_i = \int_0^{T_i} \alpha_i(f) e^{-\alpha_i(f)t} \left[ Q_i(t) + (1 - \delta_i) \left( (X_i - B_i) \frac{1 - e^{-r(T_i-t)}}{1 - e^{-rT_i}} + B_i \right) e^{-rt} \right] dt + \int_{T_i}^{\infty} \alpha_i(f) e^{-\alpha_i(f)t} Q_i(T_i) dt.$$

Using the expression for  $Q_i(t)$  in equation num, and after some algebra, we can decompose  $V_i$  into four terms:

$$V_i = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \int_0^{T_i} \alpha_i(f) e^{-\alpha_i(f)t} c_i \frac{1 - e^{-rt}}{r} dt, \\ I_2 &= (1 - \delta_i) \int_0^{T_i} \alpha_i(f) e^{-\alpha_i(f)t} \left( (X_i - B_i) \frac{1 - e^{-(r_0+y)(T_i-t)}}{1 - e^{-(r_0+y)T_i}} \right) e^{-rt} dt, \\ I_3 &= (1 - \delta_i) \int_0^{T_i} \alpha_i(f) e^{-\alpha_i(f)t} B_i e^{-rt} dt, \\ I_4 &= \int_{T_i}^{\infty} \alpha_i(f) e^{-\alpha_i(f)t} \left( c_i \frac{1 - e^{-rT_i}}{r} + B_i \right) dt. \end{aligned}$$

Where performing these integrations we get:

$$\begin{aligned} I_1 &= c \frac{1 - e^{-\alpha_i(f)T_i - \frac{\alpha_i(f)(1 - e^{-(\alpha_i(f)+r)T_i}}{\alpha_i(f)+r}}}{r} \\ I_2 &= (1 - \delta_i) \frac{\alpha_i(f) e^{-(\alpha_i(f)+r)T_i} (B_i - X_i) (-\alpha_i(f) e^{(\alpha_i(f)+r)T_i} - r e^{(\alpha_i(f)+r)T_i} + r_0 e^{T_i(r_0+y)})}{(1 - e^{T_i(r_0+y)}) (\alpha_i(f) + r) (\alpha_i(f) + r - r_0 - y)}, \\ I_3 &= (1 - \delta_i) \frac{\alpha_i(f) e^{-(\alpha_i(f)+r)T_i} (B_i - X_i) (e^{T_i(\alpha_i(f)+r+r_0+y)} (\alpha + r + r_0 + y) + y e^{T_i(r_0+y)})}{(1 - e^{T_i(r_0+y)}) (\alpha_i(f) + r) (\alpha_i(f) + r - r_0 - y)}, \\ I_4 &= (1 - \delta_i) \frac{\alpha_i(f) B_i e^{-(\alpha_i(f)+r)T_i} (e^{(\alpha_i(f)+r)T_i} - 1)}{\alpha_i(f) + r} + \frac{e^{-(\alpha_i(f)+r)T_i} (c_i (e^{rT_i} - 1) + B_i r)}{r}. \end{aligned}$$

We thus have our final expression for the value of a loan at the credit horizon if it has not already matured or defaulted. This is the value we will be needed in the Monte-Carlo simulation

## CHAPTER 6

### Semi Analytic Approximation

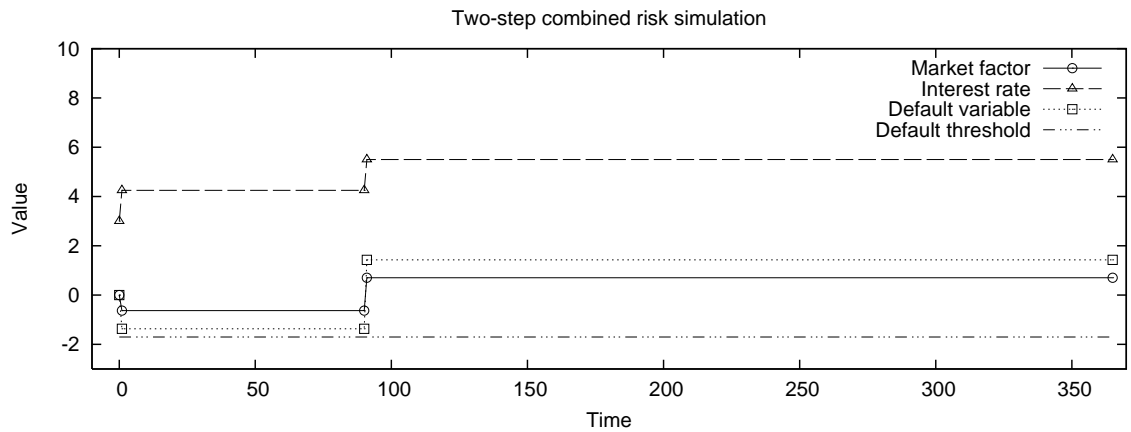
#### 6.1 The Simplified Model

We now consider a semi-analytic approximation to obtaining the loss distribution for the combined credit and interest rate risk model. The motivations are to obtain the result more efficiently and to obtain a smooth curve for purposes of estimating sensitivities, etc. To do so we will introduce several approximations to the model. We will show that those approximations are reasonable. We simplify the combined credit and interest rate risk model introduced earlier to make the analytical formulations tractable. The semi-analytic approach has several advantages over the direct Monte-Carlo method. The semi-analytic approach is deterministic and doesn't depend on the quality of a random number generator; also the semi-analytic approach can be used to estimate portfolio values, conditioning on scenarios for the market factors and the interest rate.

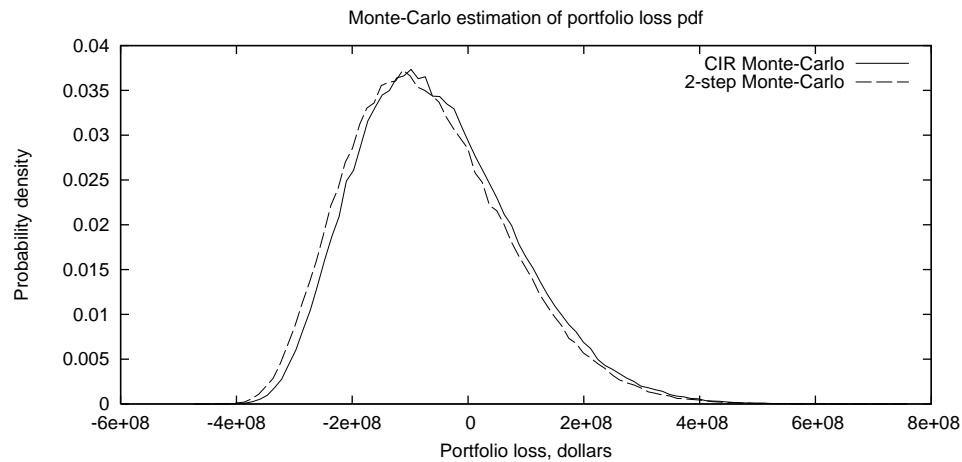
Let us first describe the simplified model. In the credit risk time horizon we have two consecutive time periods:

- 1) The time period when interest rate changes  $[0, t_{ir}]$  (usually 90 days)
- 2) The time period up to the credit horizon when interest rate is fixed  $(t_{ir}, t_{cr}]$  (usually an additional 275 days)

These time periods are typically the horizons over which one “manages” interest rate risk and credit risk respectively. We restrict the default times for the loans to the middle points of those two time periods. So the loan may default either at the midpoint of the first time-period, or at the midpoint of the second time-period. In our Monte-Carlo based model a loan could default on any day in the credit horizon, however in this simplified model the loan can default at two time points only. So for the case of default we model the values of default variables for the two time points: the end of the first period and the end of the second time period. We conducted extensive Monte-Carlo experiments to see if the simplified model is accurate comparing to the full model. We found out that the results from the simplified model



**Figure 6.1: Two step model**



**Figure 6.2: Comparison of pdfs between Monte-Carlo and the two-step Monte-Carlo**

match closely the results from the full model. So we conclude that the simplified model provides sufficient accuracy for risk management purposes (See figure 6.2).

If the value of the default variable at the end of the first time-period lies below the default threshold, we assume that the loan is defaulted during the first time period. Also we can have the event when at the end of a time period the value of the default variable is above the default threshold, but the default variable could have gone below the default threshold during the time period and then risen above the threshold before the end of the time period. We can calculate the probability

of that default in this event using a Brownian bridge:

$$pd_{BB}(l, c, \sigma, T) = e^{-\frac{2l(l+c)}{\sigma^2 T}}.$$

where  $l$  is a threshold value, 0 is the starting value of the default variable,  $c$  is the end value of the default variable,  $T$  is the length of the time period,  $\sigma$  is the standard deviation (volatility) of the default variable.

In order to calculate the loan portfolio value distribution, we will use the fact that conditioned on the market factor  $f$  and interest rate  $r$  the loans are independent. To apply the Central Limit Theorem we need to calculate the expected value and variance for each loan given the values of market factor at the ends of two time periods ( $f_{ir}$  and  $f_{cr}$ ), and given the value of interest rate at the end of the first time period ( $r_{ir}$ ). In order to calculate the expected value and the variance of a loan we need to calculate the probabilities of default during the first and the second time periods, and the value of the loan given the default in the first time period, the value of the loan given the default in the second time period and the value of the loan given no default during the simulation period. Let  $\mathcal{D}_1$  be the event that the loan defaults in the first period,  $\mathcal{D}_2$  the event that the loan defaults in the second time period; and  $\mathcal{M}$  the event that the loan doesn't default in either time period. Then for example if a loan matures after the second time period than its expected value can be expressed as:

$$E[V] = E[V|\mathcal{D}_1, r_{ir}, f_1]pd_{ir} + E[V|\mathcal{D}_2, r_{ir}, f_1, f_2]pd_{cr} + E[V|\mathcal{M}, r_{ir}, f_1, f_2](1 - pd_{ir} - pd_{cr}).$$

Let  $V_{\mathcal{D}_1} = Value(loan|\mathcal{D}_1, r_{ir}, f_1)$ ,  $V_{\mathcal{D}_2} = Value(loan|\mathcal{D}_2, r_{ir}, f_1, f_2)$  and  $V_{\mathcal{M}} = Value(loan|\mathcal{M}, r_{ir}, f_1, f_2)$ . Since  $V_{\mathcal{D}_1}$ ,  $V_{\mathcal{D}_2}$  are deterministic then we have

$$E[V] = V_{\mathcal{D}_1}pd_{ir} + V_{\mathcal{D}_2}pd_{cr} + E[V_{\mathcal{M}}](1 - pd_{ir} - pd_{cr}),$$

we also have that

$$E[V^2] = V_{\mathcal{D}_1}^2pd_{ir} + V_{\mathcal{D}_2}^2pd_{cr} + (1 - pd_{ir} - pd_{cr})E[V_{\mathcal{M}}^2],$$

where  $pd_{ir}$  is the loan's probability of default during the first time period,  $pd_{cr}$  is the loan's probability of default during the second time period. Let's derive all the variables we need in order to calculate the expected value and the variance of the loan.

## 6.2 Default During the First Time Period

The interest rate during this period is

$$r_m = \frac{r_0 + r_{ir}}{2},$$

and we assign the default to the midpoint of the first interval, so if  $T_i > \frac{t_{ir}}{2}$  then  $t_d = \frac{t_{ir}}{2}$  otherwise  $t_d = \frac{T_i}{2}$ . This condition is needed to ensure that the loan will not default after its maturity. The value of the loan has two parts: 1) the cashflows collected and reinvested at return  $r_m$  up to time  $t_d$  plus  $(1 - LGD)$  times the residual value at default time  $t_d$ . This residual for outstanding capital was derived in equation (5.1). Collecting all this together and after necessary derivations we have:

$$\begin{aligned} V_{\mathcal{D}_1} &= c \frac{e^{r_m t_d} - 1}{r_m} e^{r_m(t_{ir} - t_d)} e^{r_{ir}(t_{cr} - t_{ir})} e^{-r_0 t_{cr}} \\ &+ \frac{(1 - \delta)(X - (c - X(r_m + y)))}{r_m + y} (e^{(r_m + y)t_d} - 1) e^{r_m(t_{ir} - t_d)} e^{r_{ir}(t_{cr} - t_{ir})} e^{-r_0 t_{cr}}. \end{aligned}$$

Since we assign default to a single time  $V_{\mathcal{D}_1}$  is a deterministic number conditioned on default in the first period. One could get more careful and assign the default randomly or to the expected default time, but we do not consider these added complexities.

## 6.3 Default During the Second Time Period

If there is default during the second period, we again assign the default time to the midpoint of the period if  $T_i > \frac{t_{ir} + t_{cr}}{2}$  then  $t_d = \frac{t_{ir} + t_{cr}}{2}$  otherwise  $t_d = t_{ir} + \frac{T_i - t_{ir}}{2}$ . As before this condition is needed to ensure that the loan will not default after its maturity. As in the previous section, the value can be computed using equation

(5.1), so the value of the loan can be derived as:

$$V_{\mathcal{D}_2} = c \frac{e^{r_m t_{ir}} - 1}{r_m} e^{r_{ir}(t_{cr} - t_{ir})} e^{-r_0 t_{cr}} + \left( \frac{c(e^{r_{ir}(t_d - t_{ir})} - 1)}{r_{ir}} \right) + (1 - \delta) \frac{(X - (c - X(r_0 + y)))}{(r_0 + y)} (e^{(r_0 + y)t_d} - 1) e^{(t_{cr} - t_d)r_{ir}} e^{-r_0 t_{cr}}.$$

Since we assign default to a single time  $V_{\mathcal{D}_2}$  is a deterministic number conditioned on default in the second period.

## 6.4 No Default

There are three cases: if the loan matures in the first period, in this case the interest rate for cash is  $r_m = \frac{r_0 + r_{ir}}{2}$ . If the loan matures in the second time period, the interest rate for cash is  $r_m$  in the first period and  $r_{ir}$  in the second. If the loan matures after the second period we also have the residual value of the loan at the credit horizon as discussed in chapter 5.

### 6.4.1 The loan matures during the first time period

Accrued cash grows at interest rate  $r_m$  where as present valuing is with  $r_0$ . Using equation (5.1) we have the loan value:

$$V_{\mathcal{M}_1} = \left( c \frac{e^{r_m t} - 1}{r_m} + B \right) e^{r_m(t_{ir} - t)} e^{r_{ir}(t_{cr} - t_{ir})} e^{-r_0 t_{cr}}.$$

Conditioned on no default and the loan maturing in the first period, the loan's value is deterministic.

### 6.4.2 The loan matures during the second time period

In this case cash accrues interest at rate  $r_m = \frac{r_0 + r_{ir}}{2}$  in the first period and  $r_{ir}$  in the second time period. Using equation (5.1) we have the value of the loan:

$$V_{\mathcal{M}_2} = \left( c \frac{e^{r_m t_{ir}} - 1}{r_m} e^{r_{ir}(t_{cr} - t_{ir})} + \left( c \frac{e^{r_{ir}(t - t_{ir})} - 1}{r_{ir}} + B \right) e^{r_{ir}(t_{cr} - t)} \right) e^{-r_0 t_{cr}}.$$

Conditioned on no default and the loan maturing in the second period, the loan's value is deterministic.

### 6.4.3 The loan matures after the second time period

The value of the loan given no default during the simulation, if the loan matures after the simulation period consist of two parts, the value of cashflows plus accrued interest over the first time periods, which we define as  $V_s$ ; and the value of the residual loan after the credit horizon  $t_{cr}$ , which we denote by  $V_a$ .

Using equation (5.1) we derive the value of  $V_s$  which has two parts coming from the cash flows in the first two time periods:

$$V_s = \left( \frac{e^{r_m t_{ir}} - 1}{r_m} e^{r_{ir}(t_{cr} - t_{ir})} + \frac{e^{r_{ir}(t_{cr} - t_{ir})} - 1}{r_{ir}} \right) c e^{-r_0 t_{cr}}.$$

The value of the residual loan at the credit horizon was calculated in chapter 5:

$$V(v, B_i, T_i - t_{cr}, r_{ir}, \alpha_i(d_2)) = (I_1 + I_2 + I_3 + I_4) e^{-r_0 t_{cr}},$$

where  $T_i - t_{cr}$  is the adjusted maturity,  $r_{ir}$  is the prevailing interest rate and  $\alpha_i(d_2)$  is the new default rate parameter which depends on the realized default variable  $d_2$  and the default threshold. For Monte-Carlo methods we simulate  $d_2$  directly, but for this chapter we calculate the value of  $\alpha_i(d_2)$  which corresponds to expected probability of default based on distribution of  $d_2$ . Variable  $d_2$  has a normal distribution, conditioned on  $d_2 > -l$ . We sample values of  $d_2$  from normal distribution and reject values which are below or at  $-l$ . Then for simulated values of  $d_2$  we calculate the new probabilities of default based on the following formula:

$$pd_{new} = \frac{\Phi(-\lambda_i f_{cr} t_{cr} - d_2 - l)}{\sqrt{t_{cr}}} + e^{-2\lambda_i f_{cr}(d_2 + l)} \frac{\Phi(\lambda_i f_{cr} t_{cr} - d_2 - l)}{\sqrt{t_{cr}}}$$

Then we average over those values and obtain the average value of new default probability.

The value of the residual loan consists of four major parts  $I_1$ ,  $I_2$ ,  $I_3$  and  $I_4$ :

$$\begin{aligned}
I_1 &= \int_0^T ae^{-at} c \frac{1 - e^{-rt}}{r} dt = \frac{c - ce^{-aT} - \frac{ac(1 - e^{-(a+r)T})}{a+r}}{r}, \\
I_2 &= \int_0^T ae^{at} \left( (1 - \delta)(X - B) \frac{1 - e^{-(r_0+y)(T-t)}}{1 - e^{-(r_0+y)T}} e^{-rt} \right) dt \\
&= \frac{(-ae^{(a+r)T} - e^{(a+r)T}r + e^{T(r_0+y)}r_0 + e^{T(a+r+r_0+y)}(a+r-r_0-y) + e^{T(r_0+y)}y)}{(-1 + e^{T(r_0+y)})(a+r)(a+r-r_0-y)} \\
&\quad \times \frac{ae^{(-a-r)T}(B-X)(-1+\delta)}{(-1 + e^{T(r_0+y)})(a+r)(a+r-r_0-y)}, \\
I_3 &= \int_0^T ae^{-at} ((1 - \delta)Be^{-rt}) dt = -\frac{aBe^{-(a+r)T}(-1 + e^{(a+r)T})(-1 + \delta)}{a+r}, \\
I_4 &= \int_T^\infty ae^{-at} \left( c \frac{1 - e^{-rT}}{r} + Be^{-rT} \right) dt = \frac{e^{-(a+r)T}(c(-1 + e^{rT}) + Br)}{r},
\end{aligned}$$

We need to compute the expected value and variance of  $V_{\mathcal{M}_3}^2$ ,

$$V_{\mathcal{M}_3} = V_s + V_a$$

So

$$E[V_{\mathcal{M}_3}] = V_s + E[V_a]$$

and

$$E[V_{\mathcal{M}_3}^2] = V_s^2 + 2V_s E[V_a] + E[V_a^2]$$

We compute  $E[V_a]$  earlier in chapter 5 and we have  $E[V_a] = I_1 + I_2 + I_3 + I_4$ . We can also calculate the expected squared value of the residual loan at the credit horizon

$E[V_a^2]$ , which consists of seven parts ( $S_1, S_2, S_3, S_4, SC_{12}, SC_{13}$  and  $SC_{23}$ ):

$$\begin{aligned}
S_1 &= \int_0^T ae^{-at} \left( c \frac{1 - e^{-rt}}{r} \right)^2 dt \\
&= \frac{c^2 \left( 1 - e^{-aT} - \frac{2a(1 - e^{-(a+r)T})}{a+r} + \frac{a(1 - e^{-(a+2r)T})}{a+2r} \right)}{r^2}, \\
S_2 &= \int_0^T ae^{-at} \left( (1 - \delta) \left( (X - B) \frac{1 - e^{-(r_0+y)(T-t)}}{1 - e^{-(r_0+y)T}} \right) e^{-rt} \right)^2 dt \\
&= \frac{a(B - X)^2 \left( \frac{1}{a+2r} + \frac{e^{-2T(r_0+y)}}{a+2r-2r_0-2y} - \frac{2e^{-(a+2r)T(r_0+y)^2}}{(a+2r)(a+2r-2r_0-2y)(a+2r-r_0-y)} + \frac{2e^{-T(r_0+y)}}{-a-2r+r_0+y} \right)}{(-1 + e^{-T(r_0+y)})^2} \\
&\quad \times \frac{(-1 + \delta)^2}{(-1 + e^{-T(r_0+y)})^2} \\
S_3 &= \int_0^T ae^{-at} \left( (1 - \delta) B e^{-rt} \right)^2 dt \\
&= \frac{aB^2 e^{-(a+2r)T} (-1 + e^{a+2r}T) (-1 + \delta)^2}{a + 2r}, \\
S_4 &= \left( c \frac{1 - e^{-rT}}{r} + B e^{-rT} \right)^2 \int_T^\infty ae^{-at} dt = \\
&= \frac{ae^{-2rT} (c(-1 + e^{rT}) + Br)^2 \cdot \frac{e^{-aT}}{a}}{r^2}, \\
SC_{12} &= \int_0^T ae^{-at} \left( c \frac{1 - e^{-rt}}{r} \left( (1 - \delta) \left( (X - B) \frac{1 - e^{-(r_0+y)(T-t)}}{1 - e^{-(r_0+y)T}} e^{-rt} \right) \right) \right) dt \\
&= \frac{ac(B - X)(-1 + \delta) \left( r \left( -\frac{1}{(a+r)(a+2r)} + \frac{e^{-T(r_0+y)}}{(-a-2r+r_0+y)(-a-r+r_0+y)} \right) \right)}{(-1 + e^{-T(r_0+y)})r} \times \\
&\quad \times \frac{ac(B - X)(-1 + \delta) \left( e^{-(a+2r)T} \left( \frac{r_0+y}{(a+2r)(a+2r-r_0-y)} + e^{rT} \left( \frac{1}{a+r} + \frac{1}{-a-r+r_0+y} \right) \right) \right)}{(-1 + e^{-T(r_0+y)})r}, \\
SC_{13} &= \int_0^T ae^{-at} c \frac{1 - e^{-rt}}{r} (1 - \delta) B e^{-rt} dt = \\
&= \frac{aBc \left( \frac{1 - e^{-(a+r)T}}{a+r} - \frac{1 - e^{-(a+2r)T}}{a+2r} \right)}{r}, \\
SC_{23} &= \int_0^T ae^{-at} \left( (1 - \delta) \left( (X - B) \frac{1 - e^{-(r_0+y)(T-t)}}{1 - e^{-(r_0+y)T}} \right) e^{-rt} (1 - \delta) B e^{-rt} \right) dt = \\
&= \frac{aB e^{-(a+2r)T} (B - X)(-1 + \delta)^2 + a e^{(a+2r)T}}{(-1 + e^{T(r_0+y)})(a + 2r)(a + 2r - r_0 - y)} \\
&\quad + \frac{2e^{(a+2r)T} r - e^{T(r_0+y)} r_0 - e^{T(r_0+y)} y + e^{T(a+2r+r_0+y)} (-a - 2r + r_0 + y)}{(-1 + e^{T(r_0+y)})(a + 2r)(a + 2r - r_0 - y)},
\end{aligned}$$

$$E[V_a^2] = S_1 + S_2 + S_3 + S_4 + SC_{12} + SC_{13} + SC_{23}.$$

## 6.5 Computing Default Probabilities

To complete the derivation, we need to compute  $pd_{ir}(f_1)$  and  $pd_{cr}(f_1, f_2)$ . We can treat these two cases separately.

### 6.5.1 Default in the first period

End time  $\tau$  is defined as  $\tau = \min(T_i, t_{ir})$ . Let's calculate the probability of default during the first period as the sum of two parts  $pd_{ir1}$  and  $pd_{ir2}$ :

The probability of the default variable being below the default threshold at the end of the first time period is then given by normal cdf:

$$pd_{ir1} = \Phi_{\mu, \sigma^2}(-l) = \Phi_{\frac{\tau}{t_{ir}} \lambda_i f_{ir}, \sigma^2}(-l),$$

where  $-l$  is the default threshold.

We now compute the probability of default  $pd_{ir2}$  in  $[0, t_{ir}]$  given that the default variable is the default threshold at the end of the first time period.

So we can derive  $pd_{ir2}$  as:

$$\begin{aligned} pd_{ir2}(l, \lambda_i, \tau, t_{ir} | f_{ir}) &= \int_{-l}^{\infty} dc N_{\frac{\tau}{t_{ir}} \lambda_i f_{ir}, 1}(c) e^{-\frac{2l(l+c)}{\tau}} = \\ &= \frac{1}{\sqrt{2\tau\pi}} \int_{-l}^{\infty} dc e^{-\frac{(c - \frac{\tau}{t_{ir}} \lambda_i f_{ir})^2}{2\sigma^2}} e^{-\frac{2l(l+c)}{\sigma^2\tau}} = \\ &= \frac{1}{2} e^{-\frac{2lf_{ir}\lambda_i}{t_{ir}}} \cdot \left(1 - \operatorname{erf}\left(\frac{l - \frac{\lambda_i f_{ir}\tau}{t_{ir}}}{\sqrt{2\tau}}\right)\right). \end{aligned}$$

Finally we have:

$$pd_{ir} = pd_{ir1} + pd_{ir2}.$$

### 6.5.2 Default in the second period

Let's now calculate the probability of default during the second period,  $pd_{cr}$ . This probability can also be represented as a sum of two parts:

1. The probability  $p_{CR}$  that the default variable at the end of the second period is below the default threshold
2. The probability  $p_{CR_{BB}}$  that the default variable at the end of the second period is above the default threshold and the default variable went below the default threshold during the second period

The complication now is that we need to condition on no default in the first period. So we need to compute the probability of default in both periods: 1) The probability that the default variable is below the default threshold at the ends of both the first and the second period  $p_{IR \cap CR}$  2) The probability that the default variable goes below the default threshold during the first period, the default variable is above the default threshold at the end of the first period and the default variable is below the default threshold at the end of the second period  $p_{IR_{BB} \cap CR}$  3) The probability that the default variable goes below the default threshold during both time periods and the default variable is above the default threshold at the ends of both periods  $p_{IR_{BB} \cap CR_{BB}}$ . The probability of default in both time periods is just the sum of these three probabilities,

So the probability of default during the second time period can be expressed as:

$$pd_{cr} = (p_{CR} - p_{IR \cap CR} - p_{IR_{BB} \cap CR}) + (p_{CR_{BB}} - p_{IR_{BB} \cap CR_{BB}}),$$

To accommodate the second time period, set  $\tau = \min(T_i, t_{cr})$  The probability of the default variable being below the default threshold at the end of the second time period:

$$p_{CR} = \Phi_{\mu, \sigma^2}(-l) = \Phi_{\frac{\tau}{t_{cr}} \lambda_i f_{cr}, \sigma^2}(-l),$$

We now turn our attention to the other terms. We will express these probabilities in terms of bivariate normal cdf. Let us introduce additional variables which

will help us derive default probabilities:

$$\begin{aligned}
h &= \frac{-2l\mu_{ir}}{\sigma_{ir}^2}, \\
\mu_{ir} &= \lambda_i f_{ir}, \\
\sigma_{ir} &= t_{ir}, \\
\mu_{cr} &= \lambda_i \left( f_{ir} + (f_{cr} - f_{ir}) \frac{\tau - t_{ir}}{t_{cr} - t_{ir}} \right), \\
\sigma_{cr} &= \tau, \\
p &= \frac{x_s}{y_s},
\end{aligned}$$

By changing variables we transform integrals into analytic expressions with bivariate normal cumulative distribution function, which is a cumulative distribution function for two-dimensional normally distributed vector, where the first component is  $N(\mu_1, \sigma_1)$  and the second component is  $N(\mu_2, \sigma_2)$  and those components have correlation  $\rho$ . *bvn* is function which computes a cdf value for a bivariate normal, which takes the value for the first dimension, the value for the second dimension, the mean and standard deviation in the first dimension, the mean and standard deviation in the second dimension and correlation between the dimensions respectively. We use an adaptation of John Hull's implementation of bivariate normal cdf in our code. We define the probability of the event when the default variable is below the default threshold at the end of the first time period and at the end of the second time period:

After changing variables and some derivations we can express  $p_{IR \cap CR}$  using bivariate normal cdf:

$$p_{IR \cap CR} = bvn(l, l, \mu_{ir}, \sigma_{ir}, -\mu_{cr}, \sigma_{cr}, \frac{\sigma_{ir}}{\sigma_{cr}}),$$

We define the probability of the event when the default variable is above the default threshold at the end of the first time period, and goes below the default threshold during the first time period and at the end of the second time period:

$$p_{IR_{BB} \cap CR} = \frac{1}{2\pi\sigma_{ir}\sigma_{cr}\sqrt{1-\rho^2}} \int_{-l}^{\infty} dc_{ir} \int_{-\infty}^{-l} dc_{cr} U_1,$$

where  $U_1$  is defined as:

$$U_1 = \frac{2l(l + c_{ir})}{\sigma_{ir}^2} + \frac{\frac{(c_{ir} - \mu_{ir})^2}{\sigma_{ir}^2} + \frac{(c_{cr} - \mu_{cr})^2}{\sigma_{cr}^2} - \frac{2\rho(c_{ir} - \mu_{ir})(c_{cr} - \mu_{cr})}{c_{ir}c_{cr}}}{2(1 - \rho^2)}.$$

After changing variables and some analytic manipulations we can express  $p_{IR_{BB} \cap CR}$  using bivariate normal cdf:

$$p_{IR_{BB} \cap CR} = e^{h} bvn(-l, -l, \mu_{ir} - 2l, \sigma_{ir}, -\mu_{cr}, \sigma_{cr}, \frac{\sigma_{ir}}{\sigma_{cr}}).$$

Let us introduce some additional variables which will be helpful for default probability derivations:

$$\begin{aligned} x_m &= \lambda f_{ir}, \\ x_s^2 &= t_{ir}, \\ x_s &= \sqrt{x_s^2}, \\ y_m &= \lambda f_{ir} + \lambda(f_{cr} - f_{ir}) \frac{\tau - t_{ir}}{t_{cr} - t_{ir}}, \\ y_s^2 &= \tau, \\ y_s &= \sqrt{y_s^2}, \\ pp &= -p, \\ x_{sp} &= x_s, \\ y_{sp} &= y_s, \\ x_{mp} &= \frac{x_s^2(x_m - 2y_m) + x_m y_s^2}{y_s^2 - x_s^2}, \\ y_{mp} &= \frac{-x_s^2(2l + y_m) + (2(l + x_m) - y_m)y_s^2}{(x_s - y_s)(x_s + y_s)}, \\ x_{mp2} &= \frac{-(2lx_s x_s + x_m x_s x_s - 2x_s x_s y_m - 2ly_s y_s + x_m y_s y_s)}{(x_s x_s - y_s y_s)}, \\ y_{mp2} &= \frac{-(x_s x_s y_m - 2x_m y_s y_s + y_m y_s y_s)}{x_s x_s - y_s y_s}, \\ h_2 &= \frac{2(-x_s x_s(l + x_m - y_m) + ly_s y_s)(x_s x_s y_m - x_m y_s y_s)}{(x_s x_s(x_s x_s - y_s y_s)(x_s x_s - y_s y_s))}. \end{aligned}$$

We define the probability of the event when the default variable is above the

default threshold at the ends of the first and second time periods, and goes below the default threshold during the first and second time periods:

$$p_{IR_{BB} \cap CR_{BB}} = \frac{1}{2\pi\sigma_{ir}\sigma_{cr}\sqrt{1-\rho^2}} \int_{-l}^{\infty} dc_{ir} \int_{-l}^{\infty} dc_{cr} U_2,$$

where  $U_2$  is defined as:

$$U_2 = \frac{2l(l+c_{ir})}{\sigma_{ir}^2} + \frac{\frac{(c_{ir}-\mu_{ir})^2}{\sigma_{ir}^2} + \frac{(c_{cr}-\mu_{cr})^2}{\sigma_{cr}^2} - \frac{2\rho(c_{ir}-\mu_{ir})(c_{cr}-\mu_{cr})}{c_{ir}c_{cr}}}{2(1-\rho^2)} + \frac{2(l+c_{ir})(l+c_{cr})}{-\sigma_{ir}^2 + \sigma_{cr}^2}.$$

After changing variables and some analytic manipulations we can express  $p_{IR_{BB} \cap CR_{BB}}$  using bivariate normal cdf:

$$p_{IR_{BB} \cap CR_{BB}} = e^{h_2} bvn(-l, -l, x_{mp2}, x_{sp}, y_{mp2}, y_{sp}, pp).$$

## 6.6 A Loan's Expected Value and Variance

We now have all the necessary ingredients to calculate the expectation and variance the value of a loan. We note that  $Var[V] = E[V^2] - (E[V])^2$ . We divide all the loans into three classes:

Class 1. The loan matures during the first period:

$$E[V] = V_{D_1}pd_{ir} + E[V_{M_1}](1 - pd_{ir}),$$

$$E[V^2] = V_{D_1}^2pd_{ir} + E[V_{M_1}^2](1 - pd_{ir}),$$

Class 2. The loan matures during the second period:

$$E[V] = V_{D_1}pd_{ir} + V_{D_2}pd_{cr} + E[V_{M_2}](1 - pd_{ir} - pd_{cr}),$$

$$E[V^2] = V_{D_1}^2pd_{ir} + V_{D_2}^2pd_{cr} + E[V_{M_2}^2](1 - pd_{ir} - pd_{cr}),$$

Class 3. The loan matures during after the second period:

$$E[V] = V_{D_1}pd_{ir} + V_{D_2}pd_{cr} + E[V_{M_3}](1 - pd_{ir} - pd_{cr}),$$

$$E[V^2] = V_{D_1}^2 pd_{ir} + V_{D_2}^2 pd_{cr} + E[V_{M_3}^2](1 - pd_{ir} - pd_{cr}).$$

## 6.7 The Portfolio's Value Distribution

We can calculate the value of a loan given the two values of the market factor and the value of the interest rate. For a loan portfolio given the values of market factor at the end of the first period  $f_{ir}$  and at the end of the second period  $f_{cr}$  and the interest rate at the end of the first period  $r_{ir}$  all the loans are independent. So if we calculate the expected values and variances for the loans, we can use the Central Limit Theorem to obtain a normal distribution for a loan portfolio.

$$\begin{aligned}\mu(f_{ir}, f_{cr}, r_{ir}) &= \sum_i \mu_i(f_{ir}, f_{cr}, r_{ir}), \\ \sigma^2(f_{ir}, f_{cr}, r_{ir}) &= \sum_i \sigma_i^2(f_{ir}, f_{cr}, r_{ir}).\end{aligned}$$

So we can calculate the Cumulative Distribution Function of the portfolio's value:

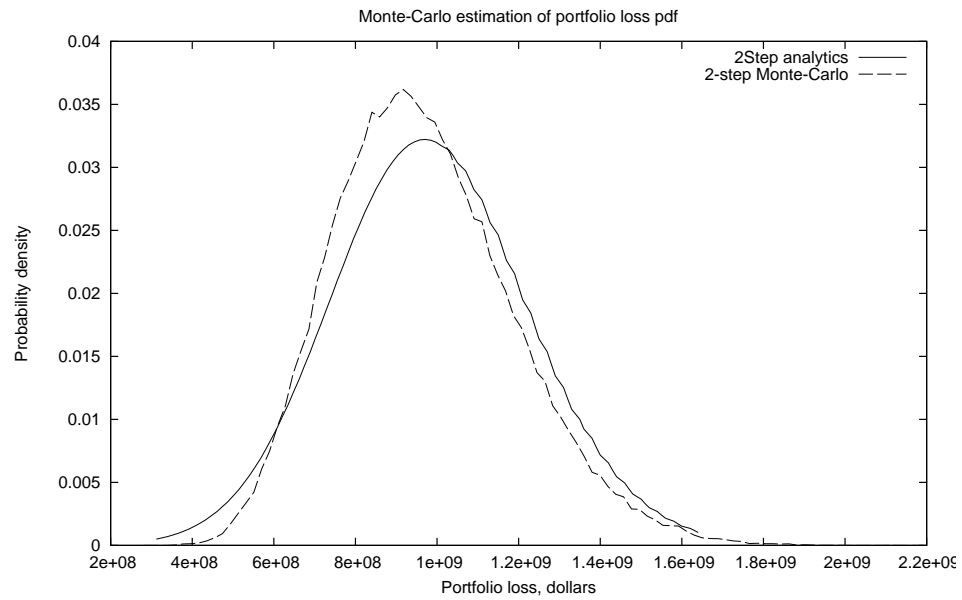
$$Pr[V \leq v | f_{ir}, f_{cr}, r_{ir}] = \Phi \left( \frac{v - \mu(f_{ir}, f_{cr}, r_{ir})}{\sigma(f_{ir}, f_{cr}, r_{ir})} \right).$$

We get the unconditional distribution by integrating

$$Pr[V \leq v] = \int df_{ir} \int df_{cr} \int dr_{ir} p(f_{ir}) p(r_{ir} | f_{ir}) p(f_{cr} | f_{ir}) \phi(v, \mu(f_{ir}, f_{cr}, r_{ir}), \sigma^2(f_{ir}, f_{cr}, r_{ir}))$$

We use numerical integration to calculate the probability that the value of the portfolio is less than the given value. We have to integrate the value of the portfolio over three dimensions:  $f_{ir}$ ,  $f_{cr}$  and  $r_{ir}$ . We use Gaussian quadrature for integration over  $f_{ir}$  and  $f_{cr}$  and equally spaced numerical integration for  $r_{ir}$ .

As we can see on Figure 6.3 there is some mismatch in pdfs between 2-step Monte-Carlo and 2-step analytics. This mismatch is caused by calculation of residual loan value. In the case of Monte-Carlo we obtain different values  $\alpha(d_2)$  by using the simulated value of default variable  $d_2$  obtaining accurate loss distribution. In the case of semi-analytic calculation, we use a single value of  $\alpha$  which is the expected value of  $\alpha$  given the distribution of  $d_2$ . We can improve the accuracy of the semi-



**Figure 6.3: Comparison of pdfs between two-step Monte-Carlo and the two-step semi-analytic solution**

analytic solution by integrating over different values of residual loan value, which will increase the running time of the solution.

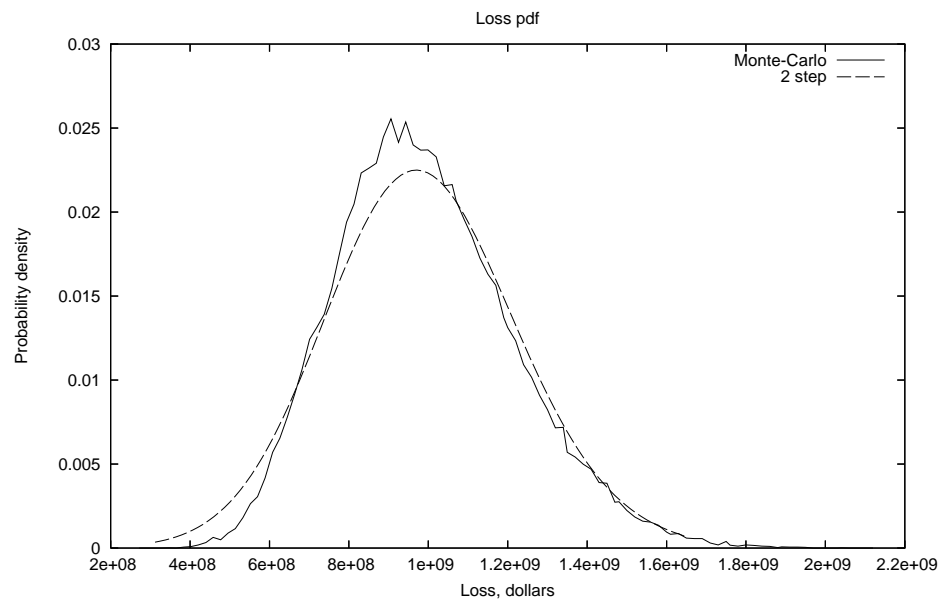


Figure 6.4: Comparison of pdfs between Monte-Carlo and the two-step semi-analytic solution

## CHAPTER 7

### Conclusion

#### 7.1 Conclusion

This work introduced a basic credit risk model for a loan portfolio. The model is factor based and the default variables of individual loans are correlated to common factors, introducing fat-tails for the loss distribution. We derived an efficient semi-analytical approximation for the model, which is suitable for very large portfolios. We showed that credit risk and interest rate risk are not independent in general and introduced a model which extends the basic credit model and takes both risks into account for portfolio loss estimation. We introduced an optimized Monte-Carlo solution method for the combined credit and interest rate model. The Monte-Carlo solution method is suitable for very large portfolios and can process portfolios with hundreds of thousands of loans on a single quad core CPU in about one hour. Also we presented a semi-analytical model for combined credit and interest rate risks. The solution for the semi-analytical approach is based on numerical integration algorithms. The model can be easily extended for multiple market factors. One of the interesting future extensions for the model is the introduction of the yield curve instead of the single interest rate which corresponds to the flat yield curve assumption.

## CHAPTER 8

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## APPENDIX

**Theorem 3.2.1** *The two central moments of loan's value distribution at time  $T$*

$$\mathbf{E}[X_i] = X_{i|\infty} - (1 - \rho_i) \int_0^\infty dz (1 - e^{-\alpha_i z}) C_i(z) P(z)$$

$$\begin{aligned} \mathbf{var}[X_i] = & (1 - \rho_i)^2 2 \int_0^\infty dz C_i(z) P(z) e^{-\alpha_i z} \int_0^z dw C_i(w) P(w) - \\ & - (1 - \rho_i)^2 \left( \int_0^\infty dz C_i(z) P(z) e^{-\alpha_i z} \right)^2 \end{aligned}$$

**Proof:** Conditioning on default in the time interval  $(t, dt)$  where  $t \leq T$ , we have

$$\begin{aligned} X_{i|t} & \stackrel{(a)}{=} \int_0^t dz C_i(z) P(z) + \rho_i \int_t^\infty dz C_i(z) P(z), \\ & \stackrel{(b)}{=} X_{i|\infty} - (1 - \rho_i) \int_t^\infty dz C_i(z) P(z). \end{aligned}$$

In (a), the first term represents the cashflows acquired up to default, and the second term the fraction of the future cashflows which can be recovered after default. In (b), we explicitly identify the loss term, where  $X_{i|\infty}$  is the value of the deal given no default. Note that since interest rate term structure is static, this is equal to the initial value of the deal.

Conditioning on no default in  $[0, T]$ , we have that

$$\begin{aligned} X_{i|\text{no def}} & \stackrel{(a)}{=} \int_0^T dz C_i(z) P(z) + \\ & + \int_T^\infty dt \alpha_i e^{-\alpha_i(t-T)} \left[ \int_T^t dz C_i(z) P(z) + \rho_i \int_t^\infty dz C_i(z) P(z) \right] \quad (3.1) \\ & = \int_0^T dz C_i(z) P(z) + \\ & + \int_T^\infty dt \alpha_i e^{-\alpha_i(t-T)} \left[ \int_T^\infty dz C_i(z) P(z) - (1 - \rho_i) \int_t^\infty dz C_i(z) P(z) \right], \\ & \stackrel{(b)}{=} X_{i|\infty} - (1 - \rho_i) \int_T^\infty dz C_i(z) P(z) (1 - e^{-\alpha_i(z-T)}). \end{aligned}$$

In (a), the first term is the payments collect to time  $T$ , and the second is the expected

payments to be collected with respect to default after  $T$ , where we have made use of the fact that the default density is memoryless.

We can proceed to obtain the full distribution of  $X_i$ . With probability  $e^{-\alpha_i T}$ ,  $X_i = X_{i|\text{no def}}$ . Otherwise, with probability  $1 - e^{-\alpha_i T}$ ,  $X_i$  takes on a value which depends on the time of default. Let  $h \in [(1 - \rho_i)X_{i|\infty}, X_{i|T}]$ .

$$\begin{aligned} \Pr [X_i \leq h] &= \Pr \left[ X_{i|\infty} - (1 - \rho_i) \int_t^\infty dz C_i(z)P(z) \leq h \right]_{f_{D_i}}, \\ &= \Pr \left[ (1 - \rho_i) \int_t^\infty dz C_i(z)P(z) \geq X_{i|\infty} - h \right]_{f_{D_i}}. \end{aligned}$$

Thus, defining  $t^*(h)$  by the integral equation

$$\int_{t^*}^\infty dt C_i(t)P(t) = \frac{X_{i|\infty} - h}{1 - \rho_i}, \quad (3.2)$$

then we have that

$$\begin{aligned} \Pr [X_i = X_{i|\text{no def}}] &= e^{-\alpha_i T}, \\ \Pr [X_i \leq h] &= \Pr [t \leq t^*]_{f_{D_i}} = \int_0^{t^*} dt \alpha_i e^{-\alpha_i t} = 1 - e^{-\alpha_i t^*(h)}, \end{aligned}$$

where  $h \in [(1 - \rho_i)X_{i|\infty}, X_{i|T}]$ . The integral equation (3.2) is generally non-trivial, however in most practical settings for  $C_i(t)$ , it can be solved in closed form. In general, the full distribution is not needed, since a portfolio with independent deals (the case under consideration) will have a value distribution which is obtained as a sum of bounded random variables<sup>1</sup>, under which scenario the strong law of large numbers applies and all we will need to construct is the mean and variance of this distribution. Nevertheless, in the event that it may be useful, the full distribution of  $X_i$  can be computed. We now consider the moments of  $X_i$ . In particular, we will

---

<sup>1</sup>Note that  $X_i$  can only take on values in  $[\rho_i X_{i|\infty}, X_{i|\infty}]$

only focus consider the loss term.

$$\begin{aligned}
\mathbf{E}[X_i] &= \int_0^T \alpha_i e^{-\alpha_i t} X_{i|t} + e^{-\alpha_i T} X_{i|\text{no def}} = & (3.3) \\
&= X_{i|\infty} - (1 - \rho_i) \int_0^T dt \alpha_i e^{-\alpha_i t} \int_t^\infty dz C_i(z) P(z) - (1 - \rho_i) e^{-\alpha_i T} \cdot \\
&\quad \cdot \int_T^\infty dz C_i(z) P(z) (1 - e^{-\alpha_i(z-T)}) = \\
&\stackrel{(a)}{=} X_{i|\infty} - (1 - \rho_i) \int_0^\infty dz (1 - e^{-\alpha_i \min\{z, T\}}) C_i(z) P(z) - \\
&\quad - (1 - \rho_i) e^{-\alpha_i T} \int_T^\infty dz C_i(z) P(z) (1 - e^{-\alpha_i(z-T)}) = \\
&= X_{i|\infty} - (1 - \rho_i) \int_0^\infty dz (1 - e^{-\alpha_i z}) C_i(z) P(z)
\end{aligned}$$

where (a) follows after reordering the integrals and performing the integral with respect to  $dt$ . To obtain  $\mathbf{var}[X_i]$ , we only need to consider the variance of the loss term. To begin, we consider the expected value of the square of the loss term in the limit  $T \rightarrow \infty$ .

$$\begin{aligned}
&(1 - \rho_i)^2 \int_0^\infty dt \alpha_i e^{-\alpha_i t} \int_t^\infty dz C_i(z) P(z) \int_t^\infty dw C_i(w) P(w), \\
&\stackrel{(a)}{=} (1 - \rho_i)^2 \int_0^\infty dz C_i(z) P(z) \int_0^z dt \alpha_i e^{-\alpha_i t} \int_t^\infty dw C_i(w) P(w), \\
&\stackrel{(b)}{=} (1 - \rho_i)^2 \int_0^\infty dz C_i(z) P(z) \int_0^\infty dw C_i(w) P(w) \int_0^{\min(w, z)} dt \alpha_i e^{-\alpha_i t}, \\
&= (1 - \rho_i)^2 \int_0^\infty dz C_i(z) P(z) \int_0^\infty dw C_i(w) P(w) (1 - e^{-\alpha_i \min(w, z)}).
\end{aligned}$$

(a) follows by interchanging the  $dt$  and  $dz$  integrals, and (b) follows from further interchanging the  $dt$  and  $dw$  integrals, and then performing the  $dt$  integration. Sub-

tracting the expected value squared, we arrive a  $\mathbf{var} [X_i]$ ,

$$\begin{aligned}
\mathbf{var} [X_i] &= (1 - \rho_i)^2 \int_0^\infty dz C_i(z)P(z) \cdot \\
&\quad \cdot \int_0^\infty dw C_i(w)P(w)(1 - e^{-\alpha_i \min(w,z)}) - (1 - e^{-\alpha_i w})(1 - e^{-\alpha_i z}) = \\
&= (1 - \rho_i)^2 \int_0^\infty dz C_i(z)P(z) \int_0^\infty dw C_i(w)P(w)(e^{-\alpha_i \max(w,z)} - e^{-\alpha_i w} e^{-\alpha_i z}) = \\
&= (1 - \rho_i)^2 \cdot 2 \int_0^\infty dz C_i(z)P(z)e^{-\alpha_i z} \int_0^z dw C_i(w)P(w) - \\
&\quad - \left( \int_0^\infty dz C_i(z)P(z)e^{-\alpha_i z} \right)^2
\end{aligned}$$

A similar calculation for bounded  $T$  yields that the expected loss squared has two components. The first is

$$e^{-\alpha_i T} (1 - \rho_i)^2 \int_T^\infty dz \int_T^\infty dw C_i(z)P(z)C_i(w)P(w)(1 - e^{-\alpha_i(z-T)})(1 - e^{-\alpha_i(w-T)}),$$

and the second is

$$\begin{aligned}
&(1 - \rho_i)^2 \int_0^T dt \alpha_i e^{-\alpha_i t} \int_t^\infty dz C_i(z)P(z) \int_t^\infty dw C_i(w)P(w), \\
\stackrel{(a)}{=} &(1 - \rho_i)^2 \int_0^\infty dz C_i(z)P(z) \int_0^{\min(z,T)} dt \alpha_i e^{-\alpha_i t} \int_t^\infty dw C_i(w)P(w), \\
\stackrel{(b)}{=} &(1 - \rho_i)^2 \int_0^\infty dz C_i(z)P(z) \int_0^\infty dw C_i(w)P(w) \int_0^{\min(w,z,T)} dt \alpha_i e^{-\alpha_i t}, \\
= &(1 - \rho_i)^2 \int_0^\infty dz C_i(z)P(z) \int_0^\infty dw C_i(w)P(w)(1 - e^{-\alpha_i \min(w,z,T)}).
\end{aligned}$$

■

**Theorem 3.4.1** *In present value dollars, we have for the first two central moments*

of the loss distribution, conditioned on  $\mathbf{f}$ ,

$$\mathbf{E}[L_i(\tau)|\mathbf{f}] = \delta'_i(Q_i(\infty, 0) - Q_i(\infty, \alpha_i)) - \gamma_i X_i, \quad (3.4)$$

$$\begin{aligned} \frac{\mathbf{var}[L_i(\tau|\mathbf{f})]}{\delta_i'^2} &= (e^{\alpha_i \tau} - 1)(Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i))^2 + \\ &+ 2Q_i(\tau, 0)(Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i)) + G_i(\tau, \alpha_i) - Q_i^2(\infty, \alpha_i), \end{aligned} \quad (3.5)$$

**Proof:** The value of the deal if no default occurs over  $[0, \tau]$  (with probability  $e^{-\alpha(\mathbf{f})\tau}$ ) is

$$V_i(\tau|\text{no default}) = Q_i(\tau) + \int_{\tau}^{\infty} \alpha_i(\mathbf{f}) e^{-\alpha_i(t-\tau)} \left[ Q_i(t) - Q_i(\tau) + \frac{1 - \delta_i}{1 + \gamma_i} (Q_i(\infty) - Q_i(t)) \right].$$

Defining  $\rho'_i = \frac{1 - \delta_i}{1 + \gamma_i}$ , we have an expression exactly analogous to (3.1), and thus we find that

$$V_i(\tau|\text{no default}) = (1 + \gamma_i)X_i - \delta'_i \int_{\tau}^{\infty} dz C_i(z) P(z) (1 - e^{-\alpha_i(\mathbf{f})(z-\tau)}),$$

where  $\delta'_i = 1 - \rho'_i = \frac{\delta_i + \gamma_i}{1 + \gamma_i}$  is the adjusted LGD. Note that the special case in which the maturity of the deal  $T_i \leq \tau$  is treated correctly – in this case,  $C_i(z) = 0$  for  $z \geq \tau$  and we correctly get  $V_i(\tau|\text{no default}) = (1 + \gamma_i)X_i$ .

Let's now consider the case that default occurs in  $[0, \tau]$ . If default occurs in  $(t, dt)$ , then we have

$$V_i(\tau|\text{default in } (t, dt)) = (1 + \gamma_i)X_i - \delta'_i(Q_i(\infty) - Q_i(t)).$$

Following the logic in Section (3.3), we arrive at

$$\mathbf{E}[V_i(\tau)] = (1 + \gamma_i)X_i - \delta'_i \int_0^{\infty} dz C_i(z) P(z) (1 - e^{-\alpha_i(\mathbf{f})z}).$$

Focussing on the loss  $L_i(\tau) = -(V_i(\tau) - X_i)$ , we have that

$$\mathbf{E}[L_i(\tau)] = \delta'_i \int_0^{\infty} dz C_i(z) P(z) (1 - e^{-\alpha_i(\mathbf{f})z}) - \gamma_i X_i.$$

Notice that the loss term in the case of no default is

$$L_i(\tau|\text{no default}) = \delta'_i \int_{\tau}^{\infty} dz C_i(z)P(z)(1 - e^{-\alpha_i(\mathbf{f})(z-\tau)}) - \gamma_i X_i,$$

and the loss term in the case of default in  $(t, dt)$  is

$$L_i(\tau|\text{default in } (t, dt)) = \delta'_i(Q_i(\infty) - Q_i(t)) - \gamma_i X_i.$$

Mimicking the analysis in Section ??, lets compute  $\mathbf{E}[L_i^2(\tau)]$ , and we will use the decomposition

$$\begin{aligned} \mathbf{E}[L_i^2(\tau)] &= \mathbf{E}[L_i^2(\tau|\text{default})] \mathbf{Pr}[\text{default}] + \mathbf{E}[L_i^2(\tau|\text{no default})] \mathbf{Pr}[\text{no default}], \\ &= \mathbf{E}[L_i^2(\tau|\text{default})] pd_i(\mathbf{f}) + \mathbf{E}[L_i^2(\tau|\text{no default})] (1 - pd_i(\mathbf{f})). \end{aligned}$$

To complete the computation, we need  $\mathbf{E}[Q_i(t)|\text{default in } (t, dt)]$  and  $\mathbf{E}[Q_i^2(t)|\text{default in } (t, dt)]$ .

$$\begin{aligned} \mathbf{E}[Q_i(t)|\text{default in } (t, dt)] &= \frac{1}{pd_i} \int_0^{\tau} dt \alpha_i e^{-\alpha_i t} \int_0^t ds C_i(s)P(s), \\ &= \frac{1}{pd_i} \int_0^{\tau} ds \int_s^{\tau} dt \alpha_i e^{-\alpha_i t} C_i(s)P(s), \\ &= \frac{1}{pd_i} \int_0^{\tau} ds C_i(s)P(s)(e^{-\alpha_i(\mathbf{f})s} - e^{-\alpha_i(\mathbf{f})\tau}). \end{aligned}$$

$$\begin{aligned} \mathbf{E}[Q_i^2(t)|\text{default in } (t, dt)] &= \frac{1}{pd_i} \int_0^{\tau} dt \alpha_i e^{-\alpha_i t} \int_0^t ds C_i(s)P(s) \int_0^t dz C_i(z)P(z), \\ &= \frac{1}{pd_i} \int_0^{\tau} ds \int_0^{\tau} dz \int_{\max(s,z)}^{\tau} dt C_i(s)P(s)C_i(z)P(z)\alpha_i e^{-\alpha_i t}, \\ &= \frac{1}{pd_i} \int_0^{\tau} ds \int_0^{\tau} dz C_i(s)P(s)C_i(z)P(z)(e^{-\alpha_i(\mathbf{f})\max(s,z)} - e^{-\alpha_i(\mathbf{f})\tau}). \end{aligned}$$

Defining a new variable,  $Y_i = L_i - \delta'_i Q_i(\infty) + \gamma_i X_i$ , which is  $L_i$  up to a constant, we

have that  $\mathbf{var} [Y_i] = \mathbf{var} [L_i]$ . It is convenient to introduce the function,

$$Q_i(t, \alpha) = \int_0^t ds C_i(s)P(s)e^{-\alpha s},$$

Note that  $Q_i(t) = Q_i(t, 0)$ , i.e.,  $Q_i(t, \alpha)$  is an extended definition of  $Q_i(t)$ . Directly from  $\mathbf{E} [L_i]$ , we have

$$\mathbf{E} [Y_i] = -\delta'_i Q_i(\infty, \alpha_i(\mathbf{f})).$$

After some algebra, we have that

$$\begin{aligned} \mathbf{E} [Y_i^2] &= \delta_i'^2 \left[ \int_0^\tau ds \int_0^\tau dz C_i(s)P(s)C_i(z)P(z)e^{-\alpha_i \max(s,z)} + 2Q_i(\tau)(Q_i(\infty, \alpha_i) - \right. \\ &\quad \left. - Q_i(\tau, \alpha_i)) + e^{\alpha_i \tau} (Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i))^2 \right]. \end{aligned}$$

Massaging this equation a little, we have

$$\frac{\mathbf{E} [Y_i^2]}{\delta_i'^2} = G_i(\tau, \alpha_i) + 2Q_i(\tau)(Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i)) + e^{\alpha_i \tau} (Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i))^2,$$

where

$$G_i(\tau, \alpha_i) = \int_0^\tau ds \int_0^\tau dz C_i(s)P(s)C_i(z)P(z)e^{-\alpha_i \max(s,z)}.$$

Finally, computing  $\mathbf{var} [L_i] = \mathbf{var} [Y_i] = \mathbf{E} [Y_i^2] - \mathbf{E} [Y_i]^2$ , we obtain

$$\frac{\mathbf{var} [L_i]}{\delta_i'^2} = e^{\alpha_i \tau} (Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i))^2 + 2Q_i(\tau, 0)(Q_i(\infty, \alpha_i) - Q_i(\tau, \alpha_i)) + G_i(\tau, \alpha_i) - Q_i^2(\infty, \alpha_i).$$

■