Pricing The American Put Using a New Class of Tight Lower Bounds

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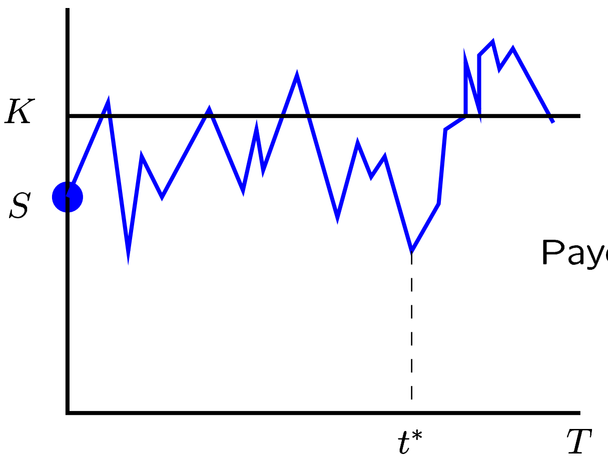
➩ **BACKGROUND**

THE BOUNDS

USING THE BOUNDS

EXPERIMENTS

The American Put Option



Payoff=max$\{0, K - S(t^*)\}$

– Continuous exercise (when to exercise?)

– What price to exercise at?

Survey

– **Data driven methods**, [Barone-Adesi87,Johnson83, Keber01,MacMillan86].

– **Discrete optimal exercise**, [Bunch92,Geske85, Huang96].

– **Bounds**, [Chen02,Levy85,Lo87].

– **Convergent numerical methods**,
Simulation, [Breen91,Broadie96,Laprise01, Longstaff01,Tsitsiklis01]
Linear Programming, [Dempster00]

For accurate pricing we need:

Continuous *and* **Optimal** exercise.

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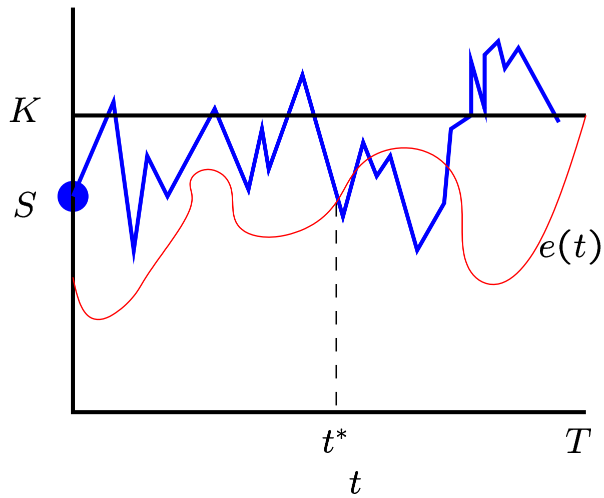
Lower Bound



Risk Neutral Measure ($\mathcal{P}$): $dS = (r - \delta)Sdt + \sigma SdW$.

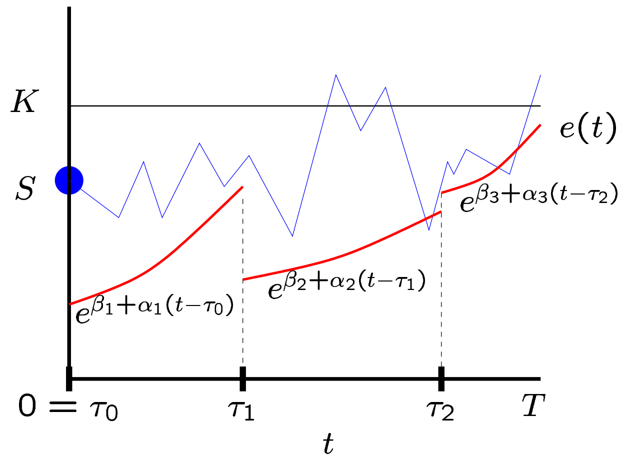
Exercise Strategy: $e(t)$.

Lower Bound:

$$\boxed{P(S, K, T, r, \delta, \sigma) \geq E_{\mathcal{P}}[Cash\ \ Flows|e(t)]}$$

for *any* $e(t)$.

Piecewise Exponential Exercise



M**−Point Bound** ($3M-1$ parameters):

$$\mathbf{L_M}(\mathbf{S},\mathbf{K},\mathbf{T}|\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}) = \sum_i a_i \mathcal{N}_{p_i}(\mathbf{x}_i, \boldsymbol{\Sigma}_i), \quad p_i \leq M.$$

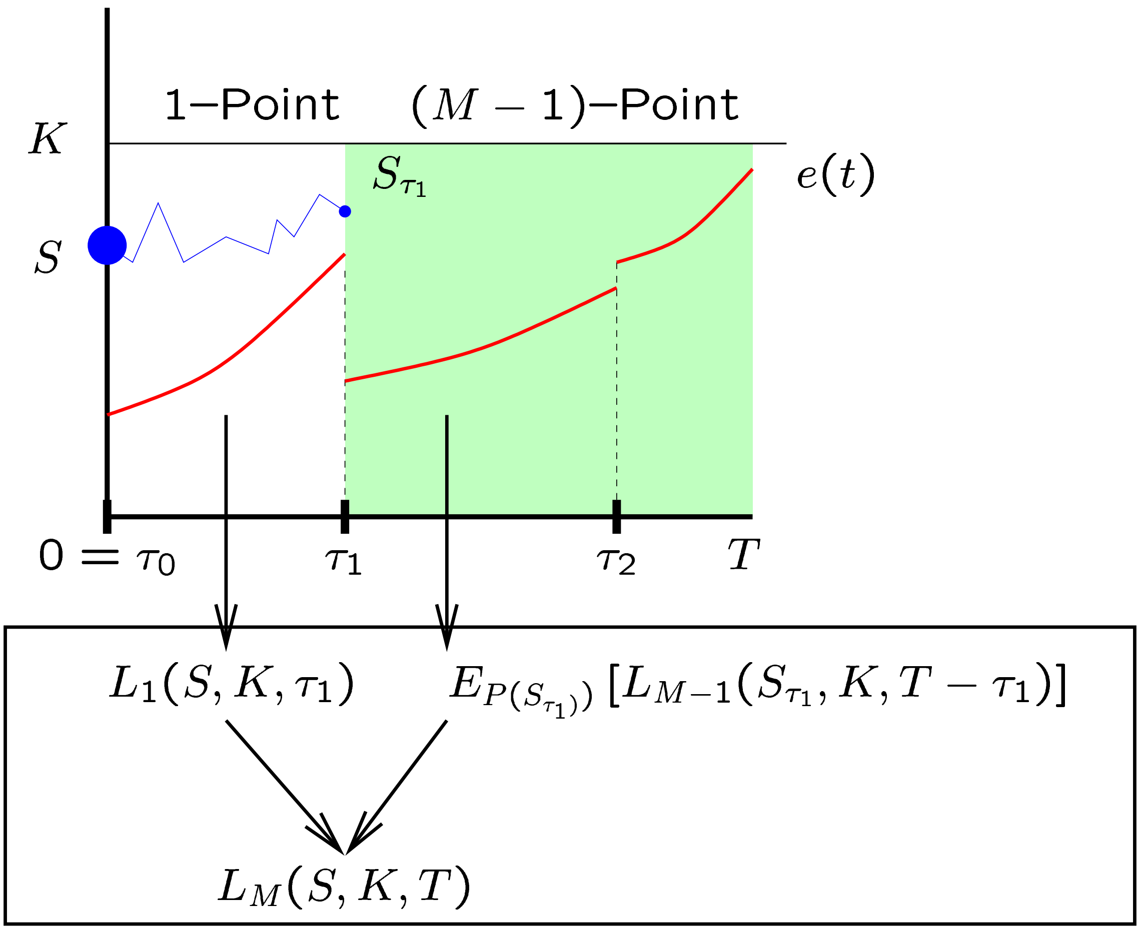
Special Cases:

Constant (0-Point Bound): $\alpha = 0, \beta$.

Exponential (1-Point Bound): α, β.

Two Piece (2-Point Bound): $\begin{Bmatrix}\beta_1\\ \alpha_1\end{Bmatrix}, \begin{Bmatrix}\beta_2\\ \alpha_2\end{Bmatrix}, \tau.$

Recursive Derivation of L_M



Analytic – can be automated!

- ✓ BACKGROUND
- ✓ THE BOUNDS
- ⇨ **USING THE BOUNDS**
- EXPERIMENTS

A Tighter Bound

– $L_M(S, K, T|\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\tau})$ is a **Lower Bound** $\forall\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\tau}\}$.

– Maximize w.r.t. $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\tau}\}$ to get a tighter bound.

Constrained Optimization Problem

$M \to \infty$, should have

$$P(S, K, T) = \sup_{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\tau}} L_M(S, K, T|\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\tau}).$$

Practical Tradeoff

$M \uparrow \implies$ more computation:

– Some M–dim multivariate Normal integrals.
– $O(12 \times 2^M)$ terms.

Small M are fast ($M = 1, 2$):

– Bivariate Normal integrals.
– Small number of terms.

- ✓ BACKGROUND
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Simulations

Tested $0, 1, 2$–Point bounds.

Canned optimization (**MATLAB**)

Ensemble of $\geq 130,000$ "randomly generated puts"

$$\{S, K, T, r, \delta, \sigma, P(S, K, T, r, \delta, \sigma)\}$$

$P(S, K, T, r, \delta, \sigma) \longleftarrow$ 15000 step binomial tree.

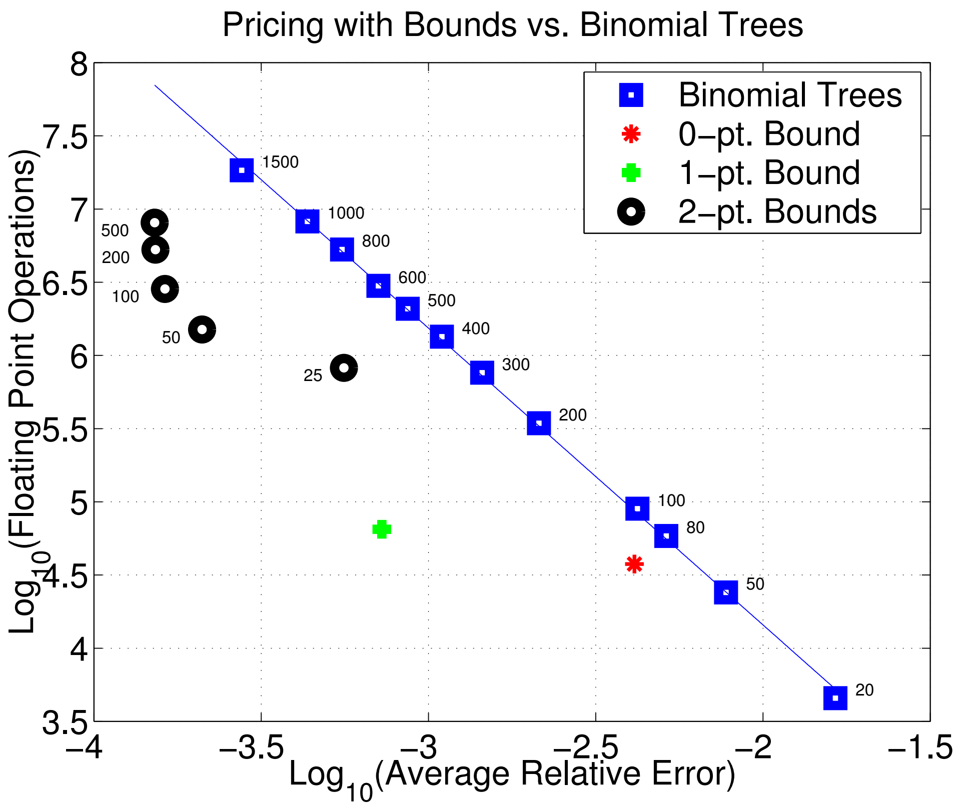
DEMO

$$\left.\begin{array}{l} S = 25 \\ K = 20 \\ T = 2.5 \\ r = 0.06 \\ \delta = 0.03 \\ \sigma = 0.2 \end{array}\right\} \longrightarrow \quad P(S, K, T, r, \delta, \sigma) = 0.6321$$

$L_0 = 0.6258$	$time \sim 0.004 sec$
$L_1 = 0.6309$	$time \sim 0.008 sec$
$L_2 = 0.6317$	$time \sim 0.142 sec$

Extensive Simulations

Comparing binomial trees to M-point Bounds.



2-point bound:

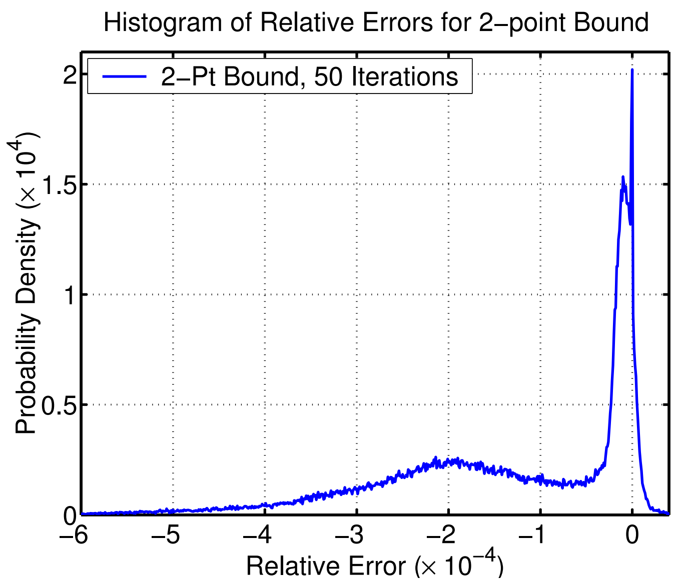
Accuracy: $10^{-3.71}$, $time \sim 0.15 sec$.

Binomial tree:

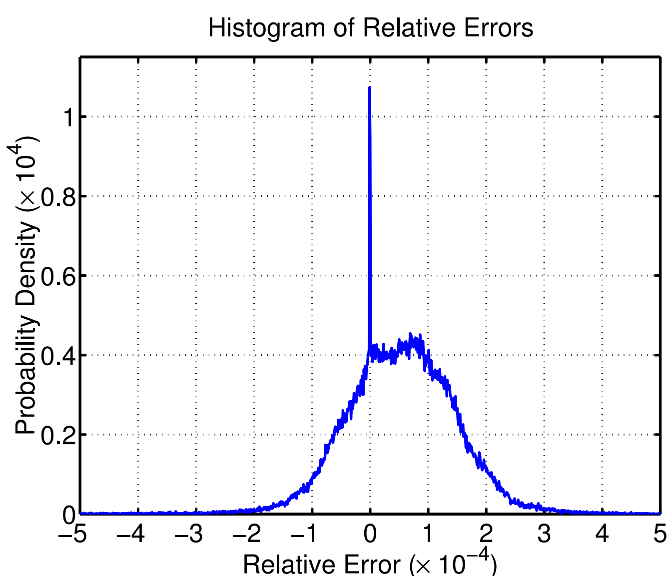
Comparable speed: $\sim \times 7$ less accurate.

Comparable accuracy: $\sim \times 25$ slower.

Histogram of Relative Errors



(a)Lower bounds.



(b)1500 step binomial tree.

Thank You

– **Family** of Analytic bounds.

– **Optimize** to get tighter bounds.

– **Ongoing:**
  Convergence?
  Data driven discrepancy prediction.
  Sensitivities.

Data available at:

www.cs.rpi.edu/∼magdon