SDE: Graph Drawing Using Spectral Distance Embedding

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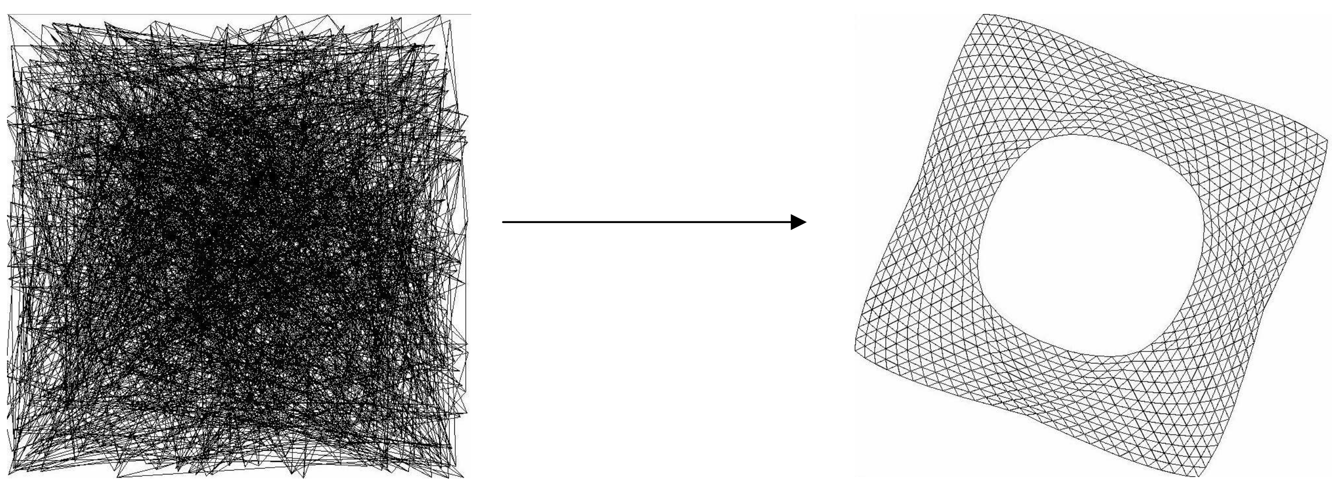
Graph Drawing Problem

- Given a graph G=(V, E), find an aesthetically pleasing layout in 2-D.
- Straight-line edge drawings of graphs: Find the coordinates of the vertices



Force-Directed Approaches

- Define an energy function or force-directed model on the graph and iterate through optimization → good results, but slow (KK ‘89, FR ‘91)
- Multi-scale approaches to make the convergence faster (HK ‘00, Walshaw ‘00)

Spectral Graph Drawing

- A new paradigm in graph drawing
- Spectral decomposition of the matrices associated with the graph
- A mathematically sound formulation of the problem preventing from iterative computation
- HDE (HK ‘02), ACE (Koren et al ‘03)

HDE (HK ’02)

- Draw the graph in high dimension (typically 50) with respect to carefully chosen pivot nodes
- Project the coordinates into two dimensions by using PCA
- Fast due to the small sizes of the matrices processed

ACE (Koren et al ’03)

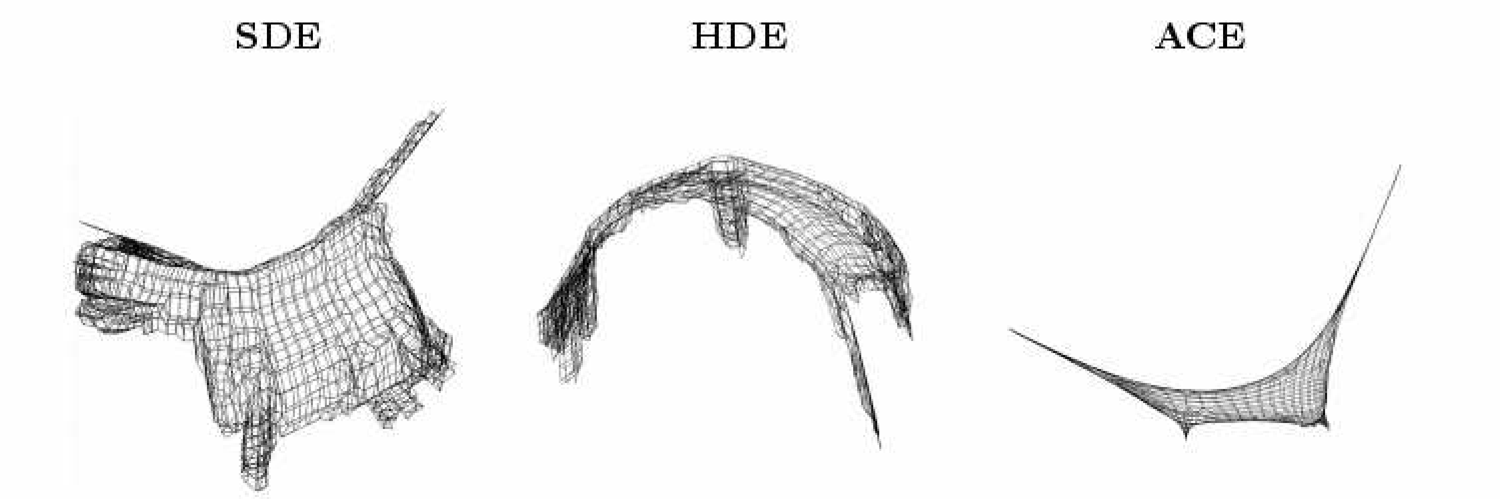
- Minimize Hall’s energy:

$$E = \frac{1}{2}\sum_{i,j=1}^{n} w_{ij}(x_i - x_j)^2$$

- Modulo some non-degeneracy and orthogonality constraints
- Eigen-decomposition of the Laplacian of the graph with a multi-scaling approach
- Fast

HDE and ACE

- Much faster than traditional force-directed methods, but with inferior aesthetics.



SDE (Spectral Distance Embedding)

- Approximate the real distances of the vertices with their graph theoretical distance
- Aesthetically pleasing drawings for a wide range of moderately large graphs with reasonable running times

Spectral Decomposition of the Distance Matrix

$$\|\mathbf{x}_i - \mathbf{x}_j\| \approx D_{ij}, \qquad \text{for } i, j = 1, 2, \ldots, n.$$

$$\mathbf{x}_i{}^2 + \mathbf{x}_j{}^2 - 2\mathbf{x}_i \cdot \mathbf{x}_j \approx D_{ij}^2.$$

- Writing in matrix notation, we have

$$\mathbf{Q}\mathbf{1}_n{}^T + \mathbf{1}_n\mathbf{Q}^T - 2\mathbf{X}\mathbf{X}^T \approx \mathbf{L}.$$

- where

$$\mathbf{X}^T = [\mathbf{x}_1, \ldots, \mathbf{x}_n], \quad \mathbf{Q}^T = [\|\mathbf{x}_1\|^2, \ldots, \|\mathbf{x}_n\|^2], \quad \mathbf{1}_n^T = [1, \ldots, 1].$$

Spectral Decomposition of the Distance Matrix

- Introducing a projection matrix

$$\gamma = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n{}^T,$$

- And letting

$$\mathbf{Y} = \gamma\mathbf{X} = (\mathbf{X} - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n{}^T\mathbf{X}).$$

- We have

$$\mathbf{Y}\mathbf{Y}^T \approx \mathbf{M} \quad = -\tfrac{1}{2}\gamma\mathbf{L}\gamma$$

Spectral Decomposition of the Distance Matrix

- $\mathbf{Y}$ is the centralized coordinates of the vertices
- Approximate $\mathbf{M}$ as closely as possible w.r.t. spectral norm
- Get the top d eigenvalues of $\mathbf{M}$

$$\mathbf{Y} = [\sqrt{\lambda_1}\mathbf{u}_1, \ldots, \sqrt{\lambda_d}\mathbf{u}_d].$$

The Algorithm

SDE(G)

1: Compute the distance matrix $\mathbf{D}$ using an APSP algorithm on G
2: Define matrix $\mathbf{L}$ such that $L_{ij} = D_{ij}^2$.
3: **return** $\mathbf{Y} = \text{PowerIteration}(-\frac{1}{2}\gamma\mathbf{L}\gamma, \epsilon)$ % epsilon is a tolerance

- Requires APSP computation which is $O(|V||E|)$
- Plus power iteration $O(d|V|^2)$
- Overall complexity: $O(|V||E| + d|V|^2)$

Performance Analysis

- Theorem: When the distance matrix is nearly embeddable and satisfies some regularity conditions, SDE recovers (up to a rotation) a close approximation to the optimal embedding
- Details in the technical report

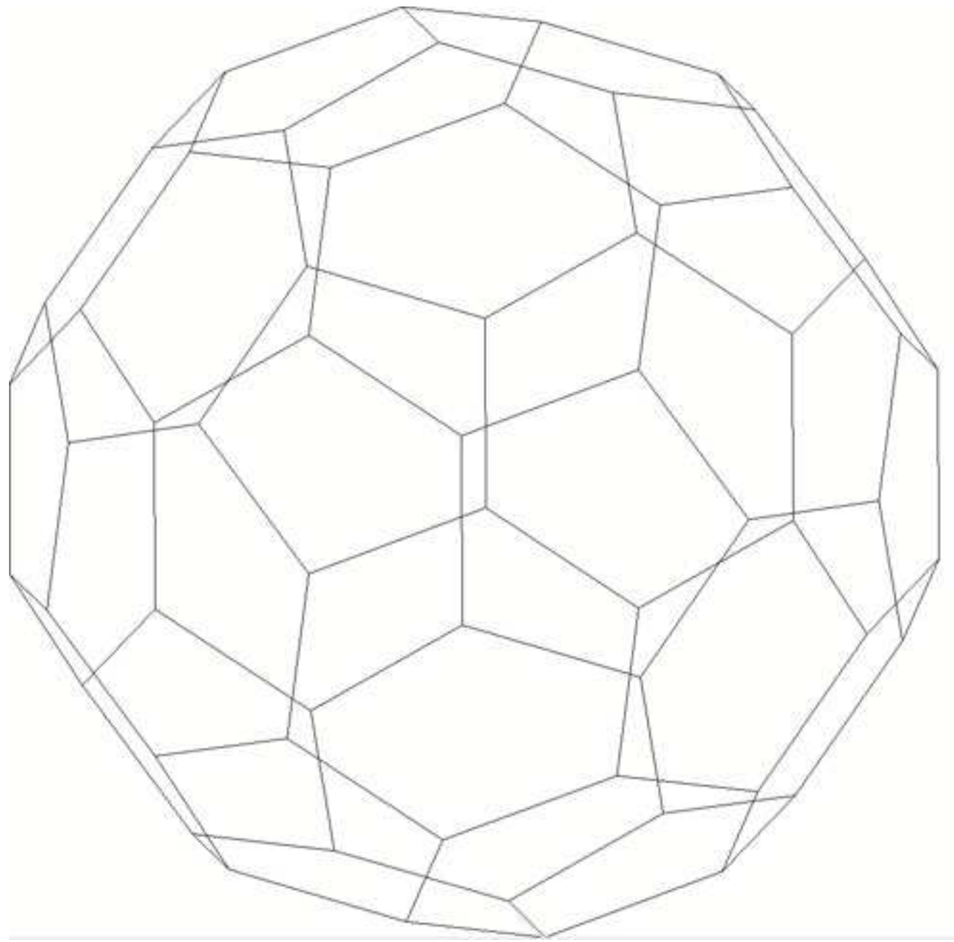
Run-times

Graph	|V|	|E|	APSP	PI	Total
Jagmesh1	936	2664	0.10	0.11	0.21
Can1072	1072	5686	0.22	0.29	0.51
Grid 50x50	2500	4900	0.65	0.54	1.19
Torus 50x50	2500	5000	0.69	0.58	1.27
Nasa1824	1824	18692	1.05	0.79	1.84
Blckhole	2132	6370	0.62	1.54	1.84
Nasa2146	2146	35052	2.07	0.69	2.76
Lshp3466	3466	10215	1.77	1.97	3.74
4970	4970	7400	2.75	2.29	5.04
Grid 70x70	4900	9660	2.80	2.43	5.23
Airfoil1	4253	12289	3.42	3.51	6.93
3elt	4720	13722	4.67	3.80	8.47

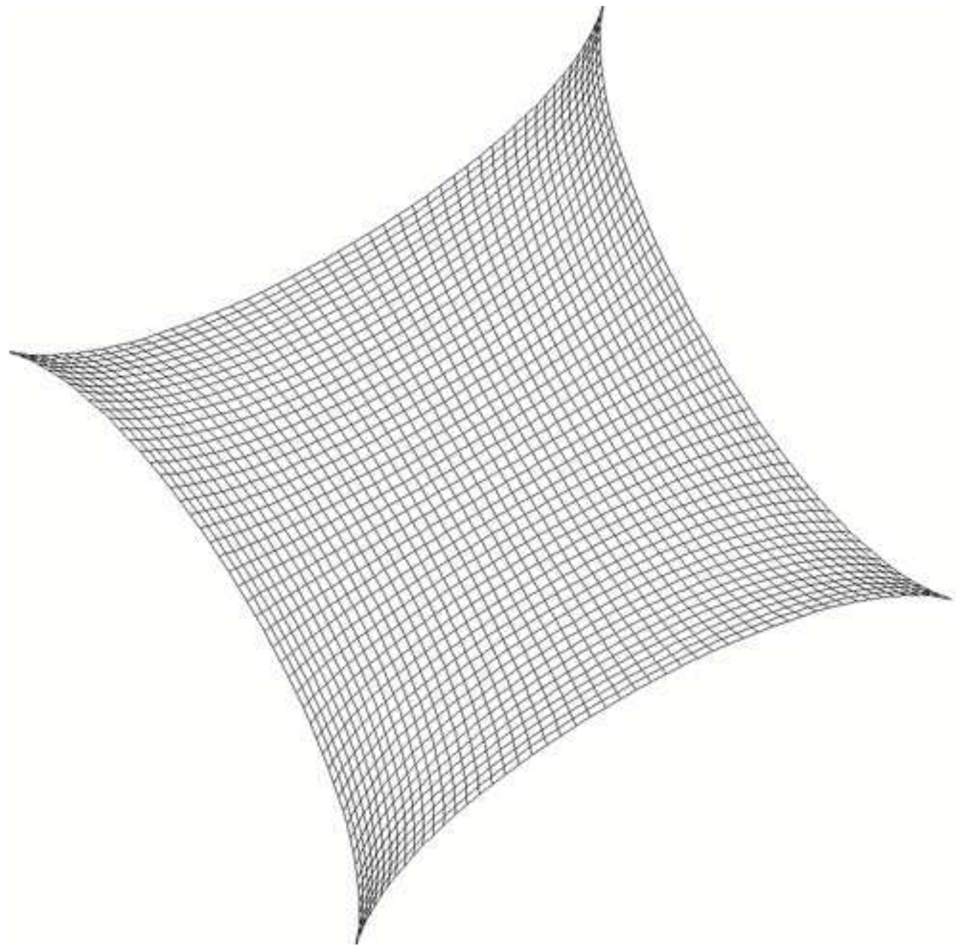
Run-times

Graph	|V|	|E|	APSP	PI	Total
Sierpinski08	9843	19683	15.71	9.01	24.72
Grid 100x100	10000	19800	18.10	11.63	29.73
Whitaker3	9800	28989	25.24	8.18	33.42
Crack	10240	30380	27.92	17.08	45.00
4elt2	11143	32818	34.35	14.42	48.77
Bcsstk33	8738	291583	76.40	15.36	91.76
4elt	15606	45878	85.81	47.52	133.33
Sphere	16386	49152	106.96	29.73	136.69
Vibrobox	12328	165250	124.51	40.30	164.81
Cti	16840	48232	91.09	77.98	169.07

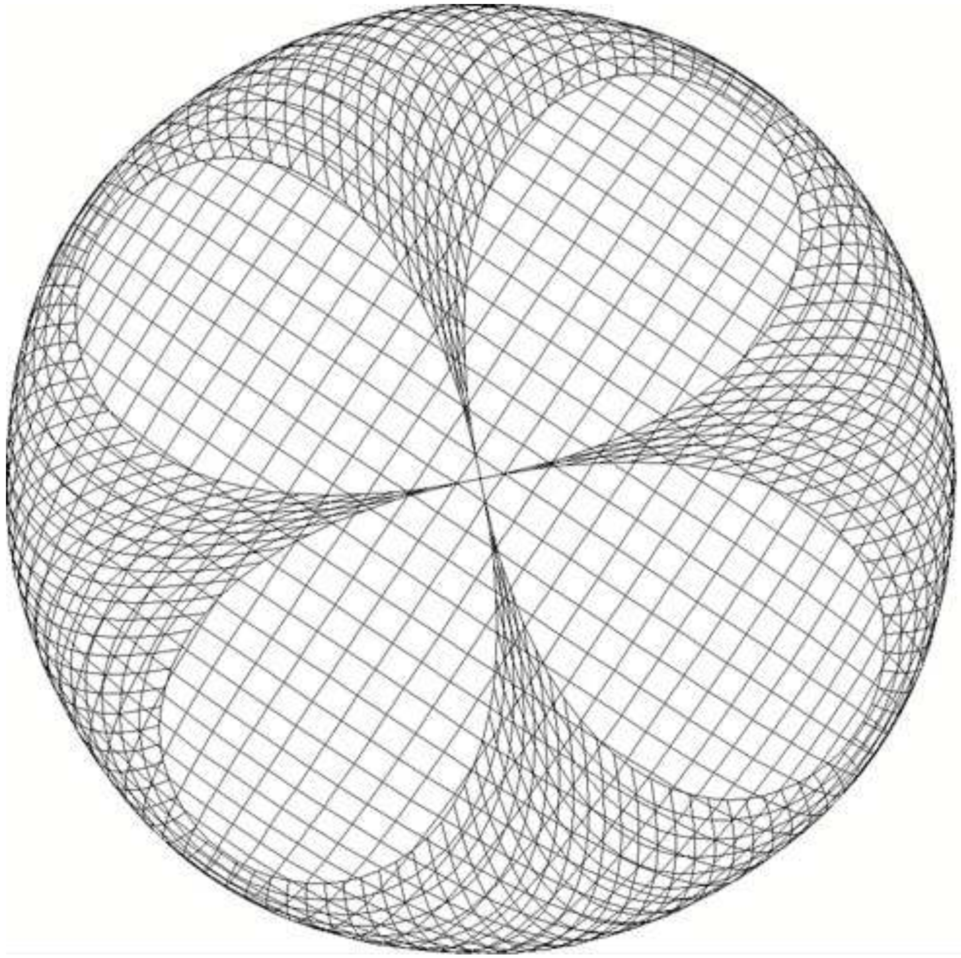
Buckyball; |V| = 60, |E| = 90



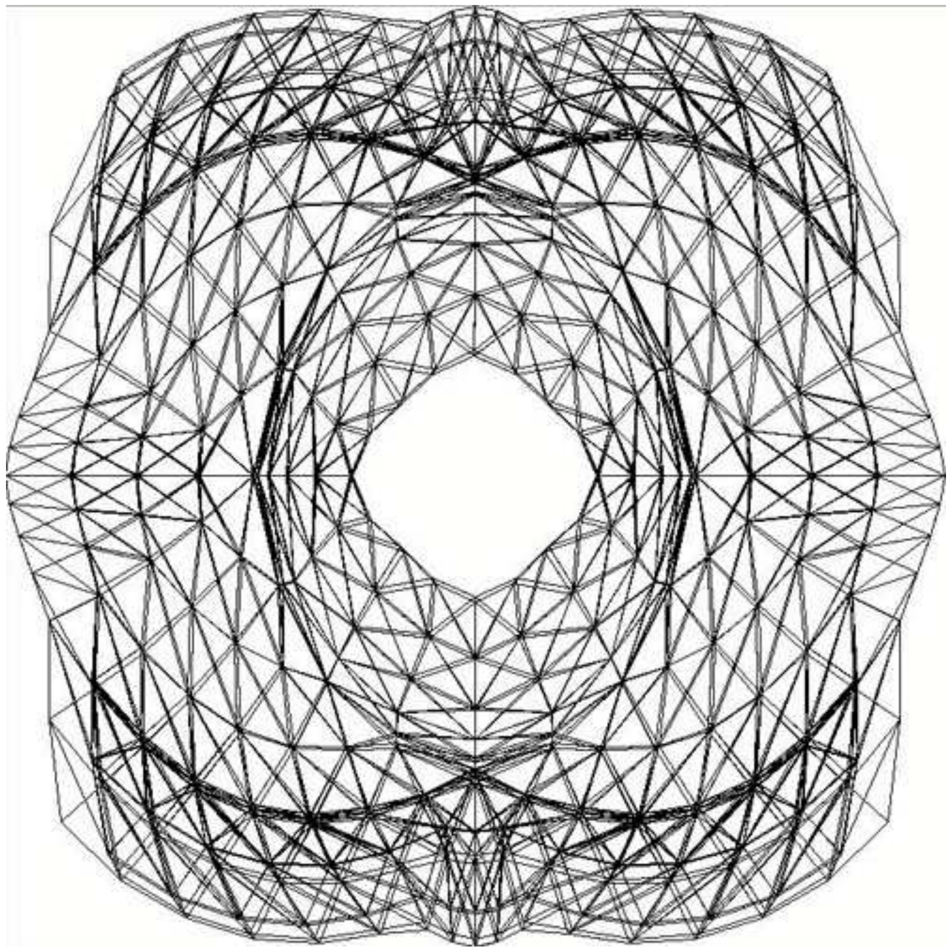
50x50 Grid; |V| = 2500, |E| = 4900



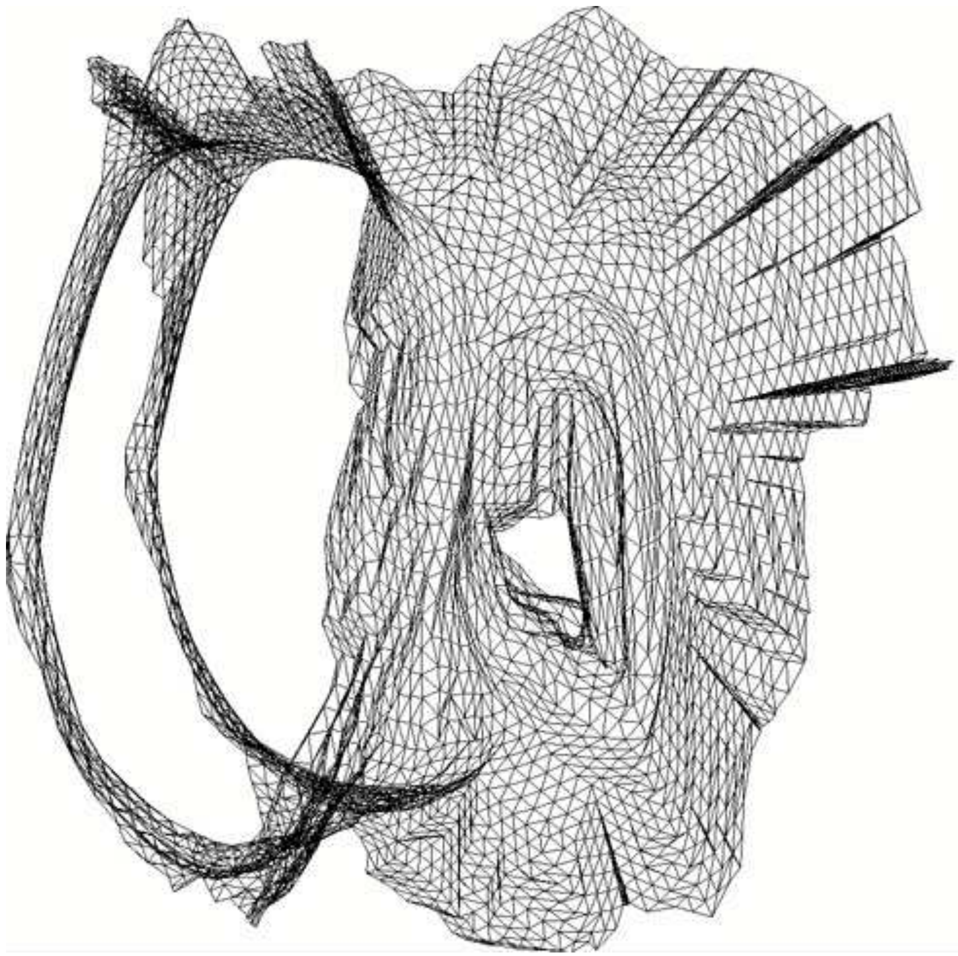
50x50 Bag; |V| = 2497, |E| = 4900



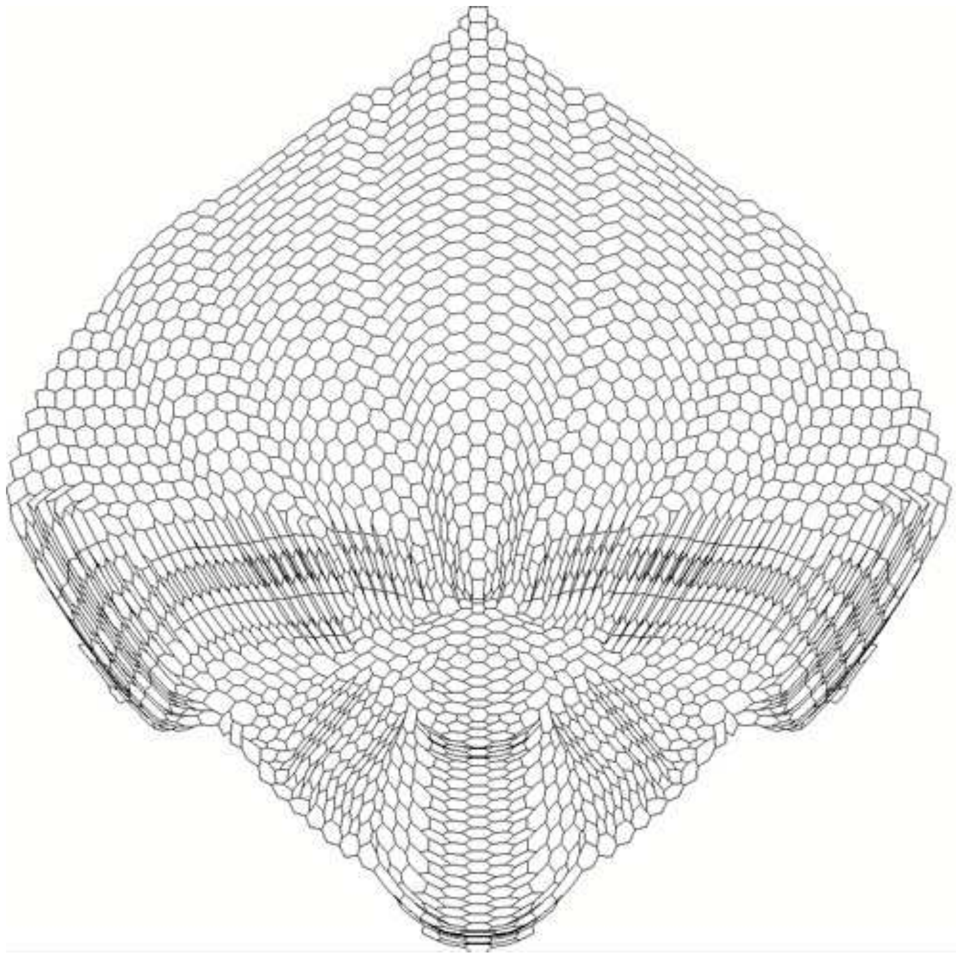
Nasa1824; |V| = 1824, |E| = 18692



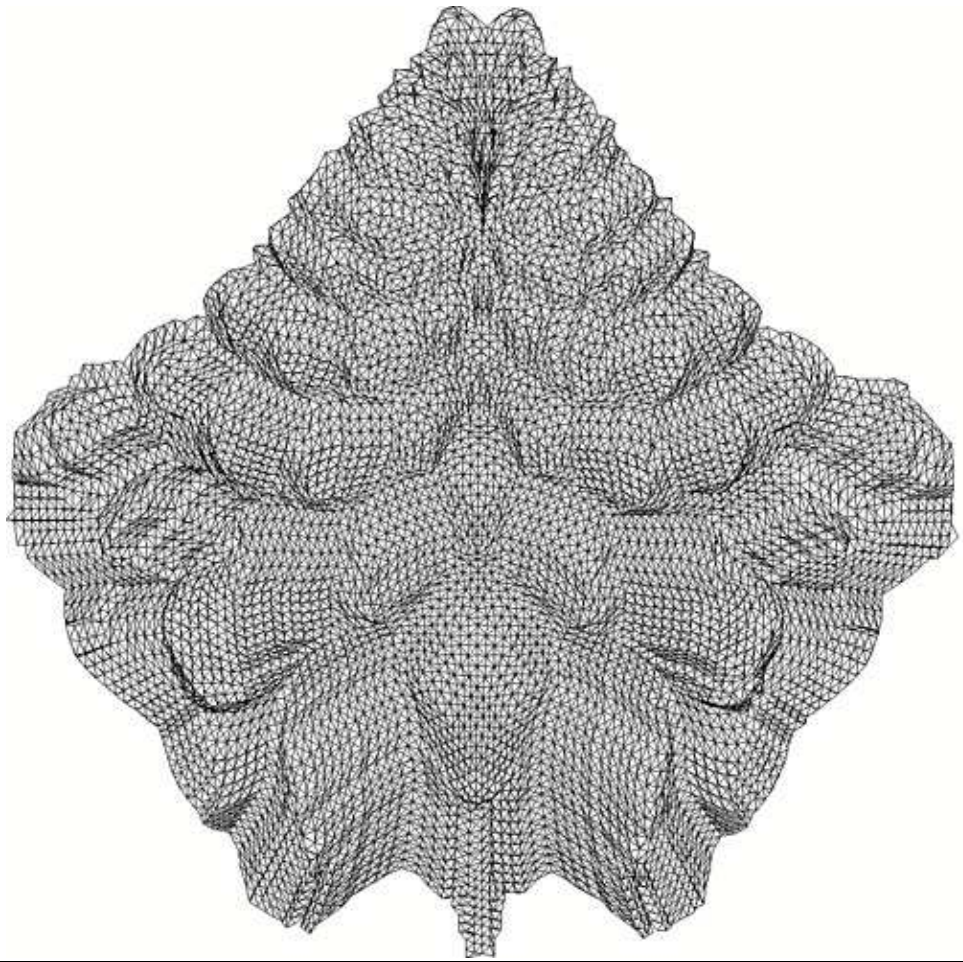
3elt; |V| = 4720, |E| = 13722



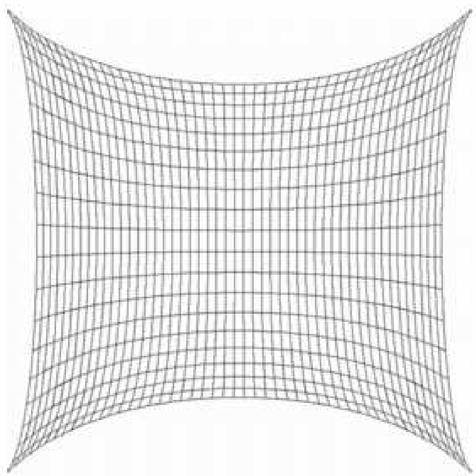
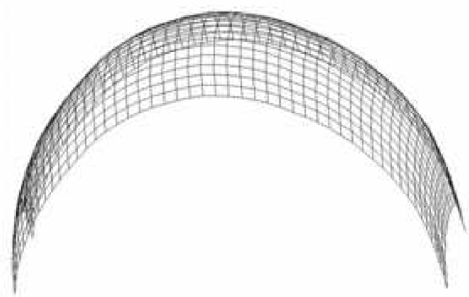
4970; |V| = 4970, |E| = 7400



Crack; |V| = 10240, |E| = 30380



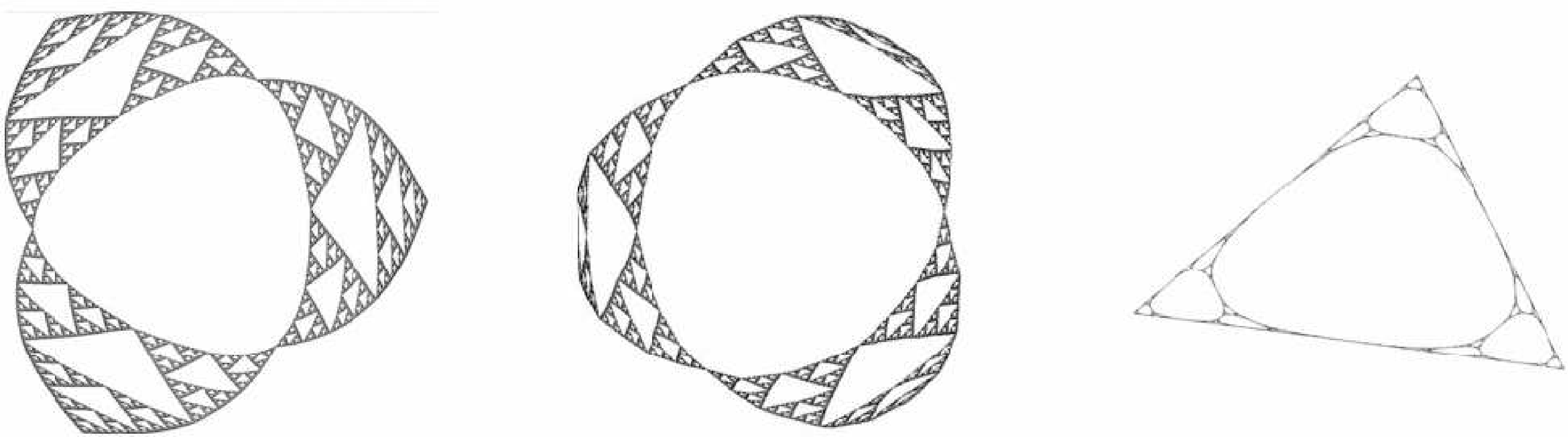
Comparison

SDE	HDE	ACE
		

20×50 grid; $|V| = 1000, |E| = 1930$.

Comparison

SDE	HDE	ACE



depth 8 sierpinski; $|V| = 9843, |E| = 19683$.

Comparison

SDE	HDE	ACE

vibrobox; $|V| = 12328, |E| = 165250.$

Conclusion

- A new spectral graph drawing method capable of nice drawings
- Reasonably fast for moderately large graphs
- Deficiencies: Cannot draw trees (as other spectral methods)

Forthcoming work

- Introduce sampling over nodes to reduce the time and space overhead
- For details, see the technical report.
- A. Civril, M. Magdon-Ismail and E. B. Rivele. SDE: Graph drawing using spectral distance embedding. Technical Report, Rensselaer Polytechnic Institute, 2005.