Using A Linear Fit To Determine Monotonicity Directions

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Abstract

We consider the problem of estimating the monotonicity directions of a function on $\mathbb{R}^d$

The idea is to estimate the monotonicity using a simple model such as a linear model. The directions implied by the linear model can then be used as constraints in a more complex model.

We show that a linear model obtains the correct monotonicity directions in many cases.

Monotonicity

A function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is said to be monotonic with direction $+1$ in dimension i if

$$f(\ldots, x_i + \Delta, \ldots) \geq f(\ldots, x_i, \ldots), \qquad (1)$$

for all $\Delta > 0$ and all $\mathbf{x} \in \mathbb{R}^d$.

We can represent the monotonicity directions of such a function by a d dimensional vector $\mathbf{m}$ of ± 1's.

There are 2^d possible choices for $\mathbf{m}$.

Some Applications

Credit Worthiness - monotonic function of income.

Medical Diagnosis - P[Heart Failure] is a monotonic function of colesterol level.

Linear Models

A linear function l is defined by

$$l(\mathbf{x}; \mathbf{w}, w_0) = \mathbf{w}^T \mathbf{x} + w_0.$$

A linear model is monotonic.

Let $p_{\mathbf{x}}(\mathbf{x})$ be the probability density for $\mathbf{x}$ with $\mathbf{\Sigma} = \int d\mathbf{x}\ p_{\mathbf{x}}(\mathbf{x})\ \mathbf{x}\mathbf{x}^T$. The L_2 linear fit (linear regression) is given by

$$\begin{aligned} \mathbf{w}^l &= \mathbf{\Sigma}^{-1} \int d\mathbf{x}\ p_{\mathbf{x}}(\mathbf{x})\ f(\mathbf{x})\mathbf{x} \\ w_0^l &= \int d\mathbf{x}\ p_{\mathbf{x}}(\mathbf{x})\ f(\mathbf{x}) \end{aligned}$$

Independent Inputs

Theorem Let $f(\mathbf{x})$ be monotonic with monotonicity direction $\mathbf{m}$. Let $\mathbf{w}^l, w_0^l$ be given by the linear fit. Then $m_i = \mathit{sign}(w_i^l)$ whenever w_i^l is non-zero.

Thus, when the inputs are independent, the linear fit deduces the correct monotonicity directions for f, even though f may not resemble a linear function in any way.

Relaxing Independence

Without independence, the linear fit may not work.

The following example establishes this fact.

$$f(\mathbf{x}) = x_1^3 - x_2$$

$$\begin{aligned} p_{\mathbf{x}}(x_1, x_2) \;=\; & \frac{1}{2} N(x_1 + 3) N(x_2 - 1) + \\ & \frac{1}{2} N(x_1 - 3) N(x_2 + 1). \end{aligned}$$

where $N(x) = e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$.

Gaussian Inputs

The Gaussian assumption is a commonly made assumption in practice.

When the inputs are Gaussian, the linear model works. In fact, the result holds for a more general class of densities defined as Mahalanobis densities.

Theorem Let $f(\mathbf{x})$ be monotonic with monotonicity direction $\mathbf{m}$. Let the input density be Mahalanobis and Let $\mathbf{w}^l, w_0^l$ be given by the linear fit. Then $m_i = \textit{sign}(w_i^l)$ whenever w_i^l is non-zero.

Finite Data

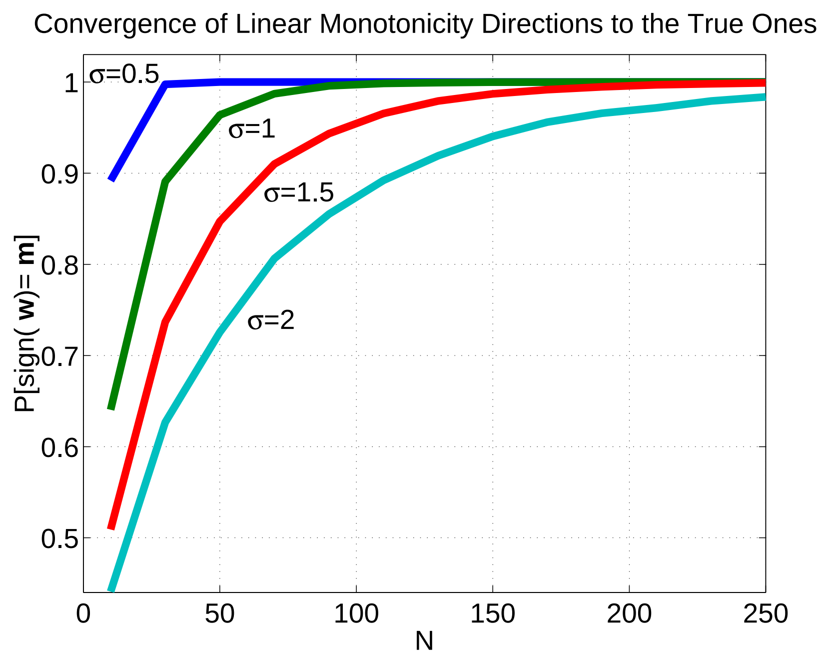
$\mathcal{D}_N = \{\mathbf{x}_i, y_i\}_{i=1}^N$

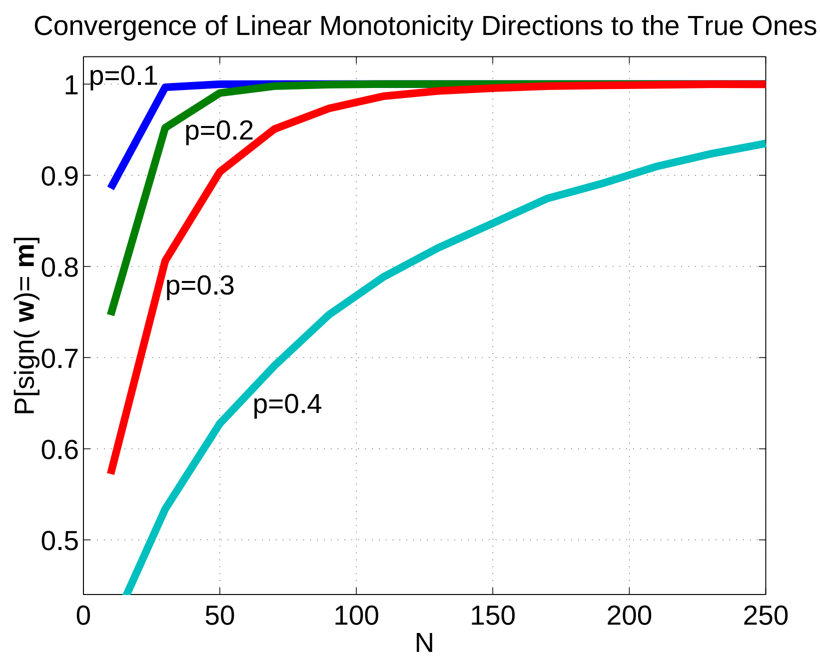
where $y_i = f(\mathbf{x}_i) + \epsilon_i$, ϵ_i are i.i.d. noise.

The Ordinary Least Squares (OLS) estimator is the sample estimate of the Linear fit.

Theorem When the linear fit obtains the correct monotonicity directions, the OLS estimator will do so asymptotically with high probability.

Some Experiments





Discussion

The linear model has a number of appealing features:

1. It is easy to implement

2. Once it has been implemented, the monotonicity directions are easy to determine

3. The linear model developed on a finite data set converges rapidly to the true linear fit

4. Our motivation is that a simple, effective algorithm be used to obtain the monotonicity directions which can then be used to *constrain* more powerful models.

Future Thought

1. Necessary and sufficient conditions for the linear model to work?

2. Expanding the class of input densities for which it works.

3. How bad can the linear model be – if the linear model gave monotonicity direction $\mathbf{m}' \neq \mathbf{m}$, and one obtains the best monotonic fit subject to the incorrect monotonicity constraints $\mathbf{m}'$, then how bad can the expected fit error be?