An Analysis of the Maximum Drawdown Risk Measure

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Motivation

Example

Fund	$\mu(\%)$	$\sigma(\%)$	max. DD(%)	T (yrs)
Π_1	25	10	-5	1
Π_2	30	10	$-7\frac{1}{2}$	1.5
Π_3	25	12.5	$-8\frac{1}{3}$	2

$$Calmar\ Ratio = \frac{\text{Return over } [0,T]}{\text{max. DD over } [0,T]}$$

$$Sterling\ Ratio = \frac{\text{Return over } [0,T]}{\text{max. DD over } [0,T] - 10\%}$$

Which fund is best?

The Complication

- μ and σ are annualized. The max. drawdown is computed over a given time period.
- Problem: funds have statistics over different length time intervals.

How do we annualize MDD?

Why?

Common Practice

Compare funds over $T = 3$ yrs.

Artificial:

- Wasteful of useful data.
- 3 years data may not be available.
- Easily available data does not generally quote the MDD for 3 years.

$\sqrt{T}$-Rule for Sharpe Ratio

$$Sharpe\ Ratio = \frac{\mu(\text{annualized})}{\sigma(\text{annualized})}$$

$$\left.\begin{array}{l}\mu_\tau \\ \sigma_\tau\end{array}\right\}\text{over time periods of size } \tau$$

$$\begin{aligned}\mu(\text{annualized}) &= \mu_\tau \cdot \frac{1}{\tau} \\ \sigma(\text{annualized}) &= \sigma_\tau \cdot \sqrt{\frac{1}{\tau}}\end{aligned}$$

$$\boxed{Sharpe\ Ratio = \frac{\mu_\tau}{\sigma_\tau\sqrt{\tau}}}$$

Similar scaling laws for Sterling-type Ratios?

Part I:

Analysis of the Maximum Drawdown.

Part II:

Application to Scaling Laws.

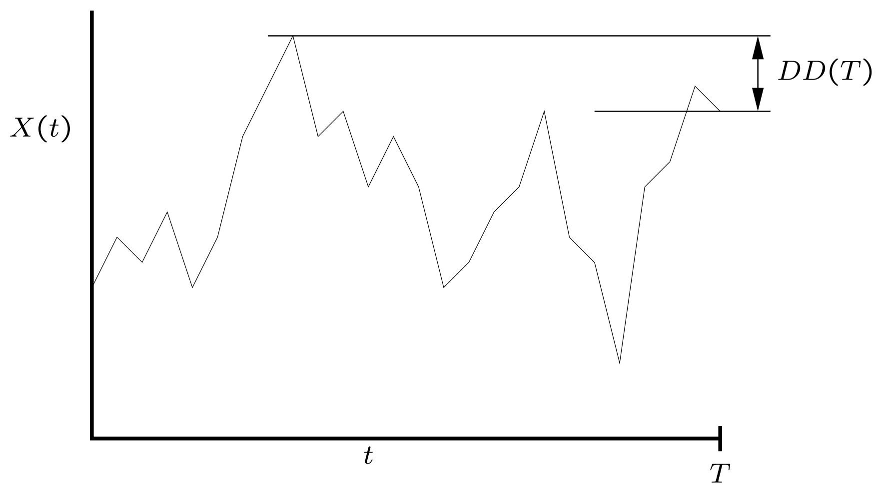
Part I:

Analysis of Maximum Drawdown

The Drawdown (DD)

The DD (current loss) is the loss from the peak to the current value.

$X(t)$ is the cumulative return curve.



$$DD(T) = \sup_{s \in [0,T]} X(s) - X(T)$$

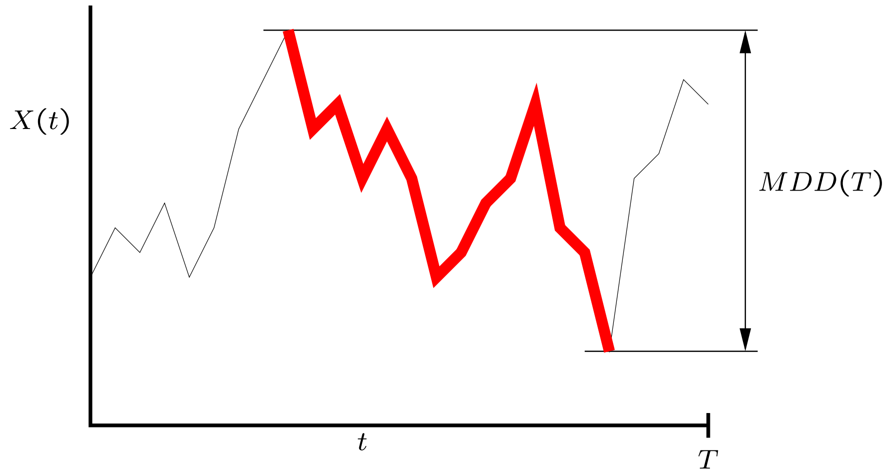
Distribution of $DD(T)$

- Since $DD(T) = \max$ over $[0, T] - X(T)$, the full distribution of $DD(T)$ can be obtained from the joint distribution of the max and the close.

- Joint distribution can be obtained in closed form for various stochastic processes (see for example **[Karatzas and Shreve, 1997]**)

The Maximum Drawdown (MDD)

The MDD is the maximum loss incurred from a peak to a bottom.



$$MDD(T) = \sup_{t \in [0,T]} DD(t)$$

– DD is an extremum.

– MDD is an **extremum of an extremum**.

Sampling of Prior Work

– Previous work on the the MDD is mostly empirical or Monte-Carlo.

- [Acar and James; 1997]
- [Sornette; 2002]
- [Burghardt, Duncan and Liu; 2003]
- [Harding, Nakou and Nejjar; 2003]
- [Chekhlov, Uryasev and Zabarankin; 2003]

– The only analytical approach is for a Brownian motion with **zero drift**, [Douady, Shiryaev and Yor; 2000].

Setup

$X(t)$ is an (arithmetic) Brownian motion:

$$\boxed{\boldsymbol{dX(t) = \mu dt + \sigma dW(t) \qquad\qquad 0 \leq t \leq T}}$$

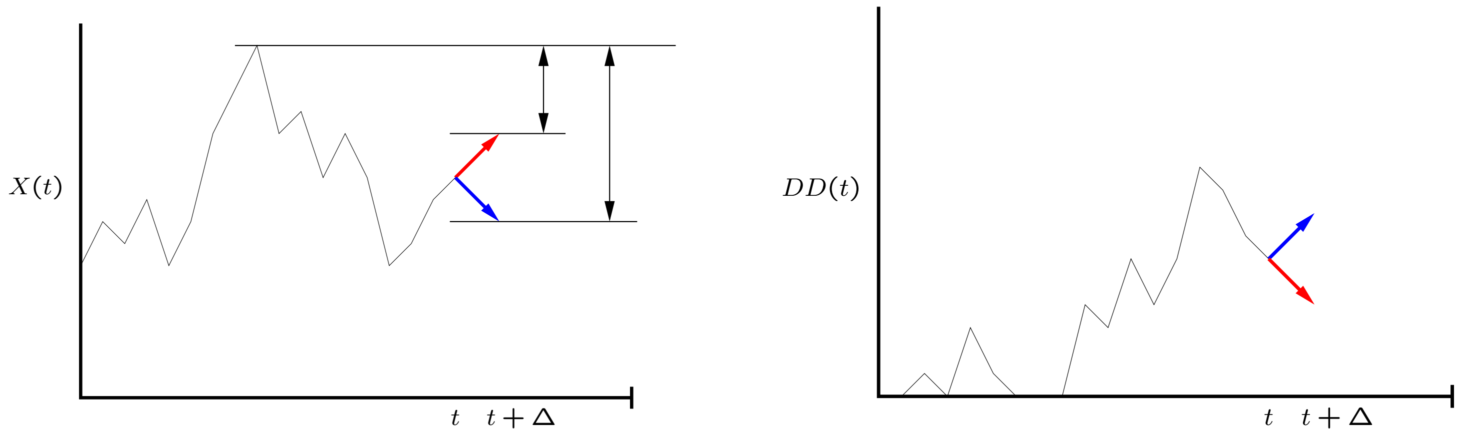
$\mu =$	average return per unit time	(drift)
$\sigma =$	std. dev. of the returns per unit time	(volatility)
$dW(t) =$	Wiener increment	(shocks)

Note: If the fund $S(t)$ follows a geometric Brownian motion, then the cumulative return sequence follows a Brownian motion.

We would like to study $MDD(\mu, \sigma, T)$.

$DD(t)$ is a Reflected Brownian Motion

Drawdown at time t is a stochastic process.

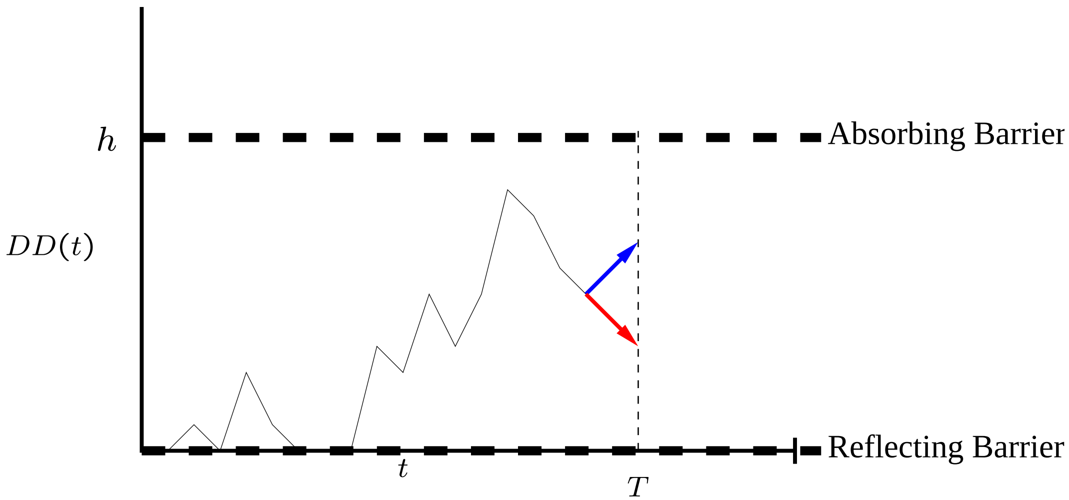


$DD(t)$ is reflecting at 0, i.e., DD(t) is a **reflected Brownian motion**.

$$dDD(t) = \begin{cases} -dX(t) & DD(t) > 0 \\ \max\left\{0, -dX(t)\right\} & DD(t) = 0. \end{cases}$$

Maximum of a Reflected Brownian Motion

$$MDD(T) = \sup_{t\in[0,T]} DD(t)$$



τ=stopping time

$f_\tau(t, h|\mu, \sigma)$=stopping time density

[Dominé; 1996]

$$G_H(h|\mu, \sigma, T) = P[MDD(\mu, \sigma, T) \geq h] = \int_0^T dt \; f_\tau(t, h| - \mu, \sigma)$$

[Magdon-Ismail, Atiya, Pratap and Abu-Mostafa; 2004]

Moments of G_H

$$E[MDD(\mu, \sigma, T)] = \int_0^\infty dh \; G_H(h|\mu, \sigma, T)$$

Expected $MDD(\mu, \sigma, T)$

Theorem: $E[MDD(\mu,\sigma,T)] = \begin{cases} \frac{2\sigma^2}{\mu} Q_p\left(\frac{\mu^2 T}{2\sigma^2}\right) & \mu > 0 \\ \sqrt{\frac{\pi}{2}}\sigma\sqrt{T} & \mu = 0 \\ \frac{2\sigma^2}{\mu} Q_n\left(\frac{\mu^2 T}{2\sigma^2}\right) & \mu < 0 \end{cases}$

Q_p and Q_n are **"universal" functions**.

– *Only need to be computed once!*

The Universal Q_p and Q_n

Bad news: Q_p and Q_n are integral series:

$$Q_p(x) = \int_0^\infty du \left[e^{-u} \sum_{n=1}^{\infty} \frac{\sin^3(\theta_n)\left(1 - e^{-\frac{x}{\cos^2(\theta_n)}}\right)}{\theta_n - \cos(\theta_n)\sin(\theta_n)} + e^{-\frac{u}{\tanh(u)}} \sinh(u) \left(1 - e^{-\frac{x}{\cosh^2(u)}}\right) \right]$$

θ_n are positive eigen solutions of $\tan(\theta_n) = \frac{\theta_n}{u}$.

$$Q_n(x) = -\int_0^\infty du\ e^u \sum_{n=1}^{\infty} \frac{\sin^3(\theta_n)\left(1 - e^{-\frac{x}{\cos^2(\theta_n)}}\right)}{\theta_n - \cos(\theta_n)\sin(\theta_n)}$$

θ_n are positive eigen solutions of $\tan(\theta_n) = -\frac{\theta_n}{u}$.

Good news: Only have to be computed once.

More Good News: Asymptotic Behavior

Most trading desks are interested in the long term.

Theorem:

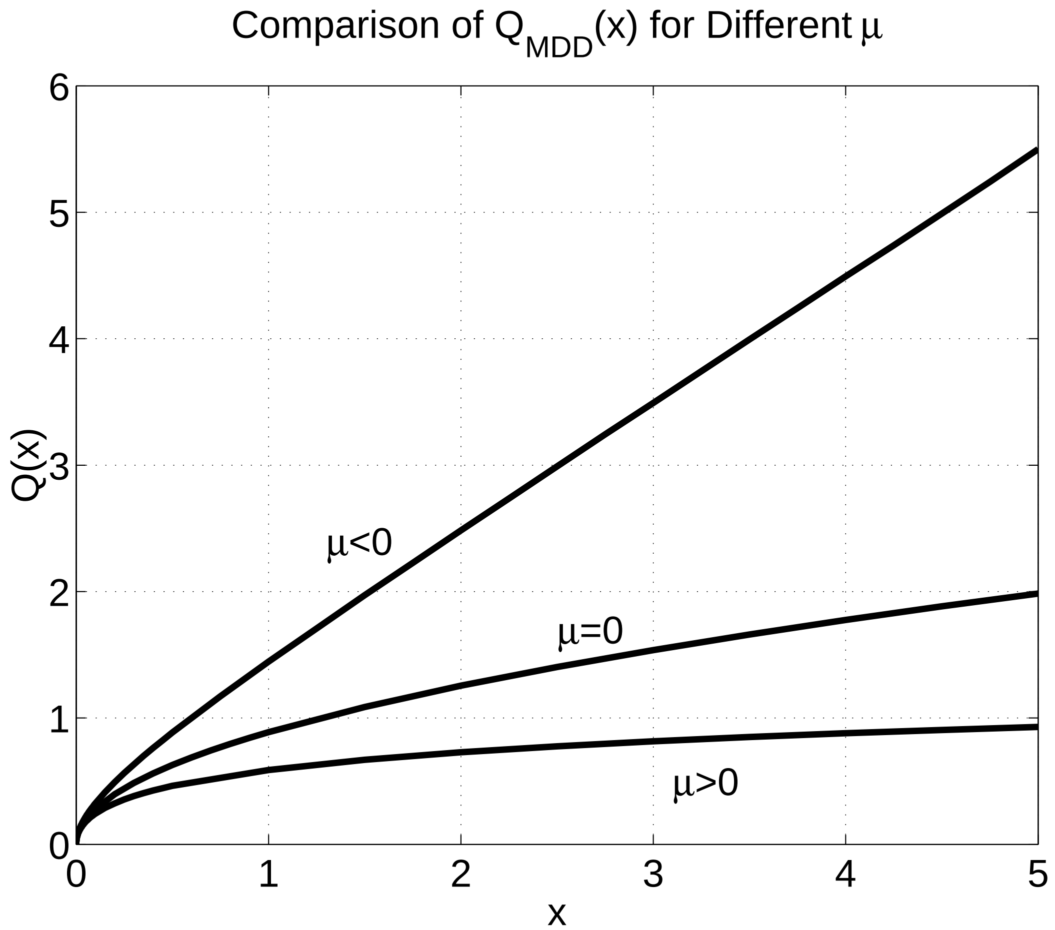
$$E[MDD] \xrightarrow{T\to\infty} \begin{cases} \frac{\sigma^2}{\mu}\left(0.63519 + \frac{1}{2}\log T + \log\frac{\mu}{\sigma}\right) & \mu > 0 \\ \sqrt{\frac{\pi}{2}}\sigma\sqrt{T} & \mu = 0 \\ \mu T + \frac{\sigma^2}{\mu} & \mu < 0 \end{cases}$$

Phase Transitions: Three different types of behavior:

$$\log T,\ \sqrt{T},\ T$$

depending on the regime of μ.

Behavior of $Q_{MDD}(x)$



http://www.cs.rpi.edu/∼magdon/data/Qfunctions.html

Some Useful Statistics

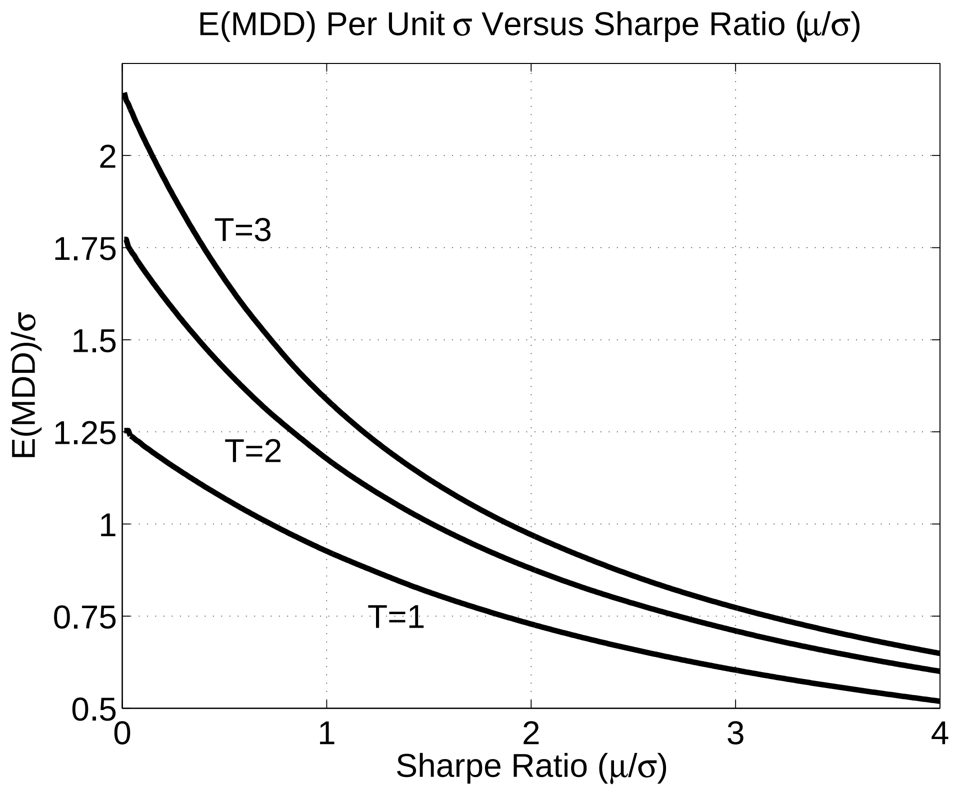
$\frac{E[MDD]}{\sigma}$ Expected maximum drawdown per unit volatility

$\text{Shrp} = \frac{\mu}{\sigma}$ Sharpe ratio of expected performance

$\text{Clmr} = \frac{\mu T}{E[MDD]}$ Calmar ratio of expected performance

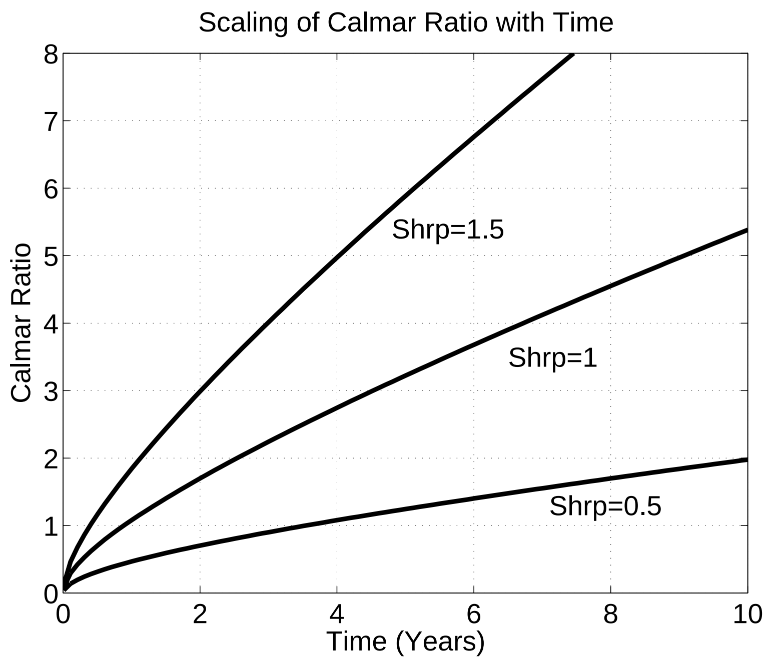
From now on, $\mu > 0$.

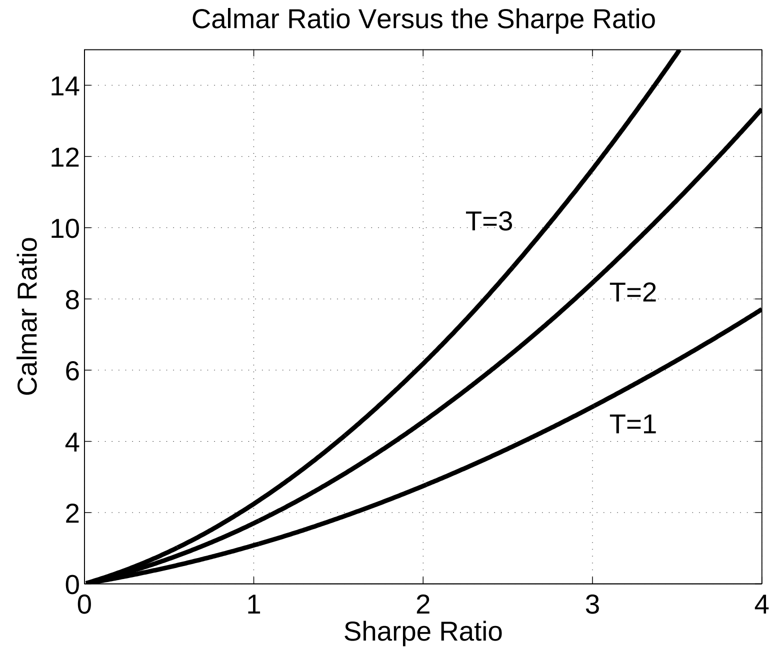
MDD **vs.** Shrp



$$\frac{E[MDD]}{\sigma} = \frac{2Q_p\left(\frac{T}{2}\text{Shrp}^2\right)}{\text{Shrp}} \xrightarrow{T\to\infty} \frac{0.63519 + 0.5\log T + \log \text{Shrp}}{\text{Shrp}}$$

Calmar vs. Sharpe





$$\mathsf{Clmr} = \frac{\frac{T}{2}\mathsf{Shrp}^2}{Q_p\left(\frac{T}{2}\mathsf{Shrp}^2\right)} \xrightarrow{T\to\infty} \frac{T\mathsf{Shrp}^2}{0.63519 + 0.5\log T + \log \mathsf{Shrp}}$$

Note: Clmr is monotonic in Shrp.

Part II:

Scaling Laws

Recap

$$\text{Calmar}(T) = \frac{\text{Return over } [0, T]}{MDD \text{ over } [0, T]} \approx \frac{\mu T}{E[MDD]} = \text{Clmr}$$

Given two funds,

$\Pi_1 : \mu_1, \sigma_1, T_1, MDD_1, \text{Clmr}_1 = \frac{\mu_1 T_1}{MDD_1}$.

$\Pi_2 : \mu_2, \sigma_2, T_2, MDD_2, \text{Clmr}_2 = \frac{\mu_2 T_2}{MDD_2}$.

How to compare Clmr_1 and Clmr_2?

Normalized Calmar Ratio I

Normalize the ratios to a reference time τ, for example $\tau = 1$ yr.

We know how to scale return over $[0, T_1]$,

$$\mu_1 T_1 \longrightarrow \mu_1 \tau = \mu_1 T_1 \cdot \frac{\tau}{T_1}$$

We can scale $MDD([0, T_1]) \rightarrow MDD([0, \tau])$ using proportion,

$$\frac{E[MDD([0, \tau])]}{E[MDD([0, T_1])]} = \frac{MDD([0, \tau])}{MDD([0, T_1])}$$

Normalized Calmar Ratio II

$$\begin{aligned}\overline{Calmar}_1(\tau) &= \frac{\text{return}([0,\tau])}{MDD([0,\tau])} \\ &= \underbrace{\frac{\mu_1 T_1}{MDD([0,T_1])}}_{Calmar_1(T_1)} \cdot \underbrace{\frac{\tau}{T_1} \cdot \frac{E[MDD([0,T_1])]}{E[MDD([0,\tau])]}}_{\gamma_\tau}\end{aligned}$$

$$\boxed{\begin{gathered}\overline{Calmar}_1(\tau) = \gamma_\tau(T_1, \mathsf{Shrp}_1) \times Calmar_1(T_1), \\ \gamma_\tau(T_1, \mathsf{Shrp}_1) = \frac{\frac{1}{T_1} Q_p(\frac{T_1}{2}\mathsf{Shrp}_1^2)}{\frac{1}{\tau} Q_p(\frac{\tau}{2}\mathsf{Shrp}_1^2)}\end{gathered}}$$

Example – Revisited

Fund	$\mu(\%)$	$\sigma(\%)$	max. DD(%)	T (yrs)	**Calmar**	$\overline{\textbf{Calmar}}$
Π_1	25	10	-5	1	**5**	**5**
Π_2	30	10	$-7\frac{1}{2}$	1.5	**6**	**4.41**
Π_3	25	12.5	$-8\frac{1}{3}$	2	**6**	**3.62**

(normalized to $\tau = 1$ yr.)

$$\boxed{\Pi_1 > \Pi_2 > \Pi_3}$$

τ-Relative Strength

Fix a reference time τ.

$$\beta_\tau(\Pi_1|\Pi_2) = \frac{\overline{Calmar}_1(\tau)}{\overline{Calmar}_2(\tau)}.$$

– Relative normalized Calmar ratio of Π_1 with respect to Π_2.

– May depend on τ.

Relative Strength

Consider the long horizon, $\tau \to \infty$,

$$\beta(\Pi_1|\Pi_2) = \lim_{\tau\to\infty} \beta_\tau(\Pi_1|\Pi_2)$$

– The limit exists.

$$\textit{relative strength} = \beta(\Pi_1|\Pi_2) = \frac{Calmar_1(T_1)}{Calmar_2(T_2)} \times \frac{\frac{1}{T_1} Q_p(\frac{T_1}{2}\mathsf{Shrp}_1^2)}{\frac{1}{T_2} Q_p(\frac{T_2}{2}\mathsf{Shrp}_2^2)}$$

$$\Pi_1 \succeq \Pi_2 \iff \beta(\Pi_1|\Pi_2) \geq 1$$

Example – Re-Revisited

Fund	$\mu(\%)$	$\sigma(\%)$	max. DD(%)	T (yrs)	β
Π_1	25	10	-5	1	**1.00**
Π_2	30	10	$-7\frac{1}{2}$	1.5	**0.97**
Π_3	25	12.5	$-8\frac{1}{3}$	2	**0.64**

(relative strengths w.r.t Π_1.)

$$\boxed{\Pi_1 > \Pi_2 > \Pi_3}$$

Properties of Relative Strength

Complete: $\beta(\Pi_1|\Pi_2) = \frac{1}{\beta(\Pi_2|\Pi_1)}$

$$\boxed{\Pi_1 \succeq \Pi_2 \text{ or } \Pi_2 \succeq \Pi_1}$$

Transitivity: $\beta(\Pi_1|\Pi_3) = \beta(\Pi_1|\Pi_2)\beta(\Pi_2|\Pi_3)$

$$\boxed{\Pi_1 \succeq \Pi_2 \text{ and } \Pi_2 \succeq \Pi_3 \implies \Pi_1 \succeq \Pi_3}$$

Independent of Reference instrument: $\beta(\Pi_1|\Pi_2) = \frac{\beta(\Pi_1|\Pi_3)}{\beta(\Pi_2|\Pi_3)}$

$$\boxed{\beta(\Pi_1|\Pi_3) \geq \beta(\Pi_2|\Pi_3) \implies \Pi_1 \succeq \Pi_2}$$

i.e., the relative strength defines a **total order**.

Real Data*

Fund	μ(%)	σ(%)	T(yrs)	MDD	$Calmar$	**E[MDD]**	$\overline{\textbf{Calmar}}$	$\boldsymbol{\beta}$
$S\&P500$	10.04	15.48	24.25	46.28	5.261	**44.56**	**0.6104**	**1**
$FTSE100$	7.01	16.66	19.83	48.52	2.865	**55.54**	**0.4395**	**0.5003**
$NASDAQ$	11.20	24.38	19.42	75.04	2.899	**77.87**	**0.4402**	**0.5407**
DCM	15.65	5.78	3.08	3.11	15.50	**4.770**	**6.541**	**27.76**
NLT	3.35	16.03	3.08	25.40	0.4062	**31.35**	**0.2202**	**0.1331**
OIC	17.19	4.52	1.16	0.42	47.48	**2.493**	**42.31**	**212.0**
TGF	8.48	9.83	4.58	8.11	4.789	**15.84**	**1.752**	**3.589**

DCM=Diamond Capital Management;
NLT=Non-Linear Technologies;
OIC=Olsen Investment Corporation;
TGF=Tradewinds Global Fund.

– Normalized Calmar ratio is to $\tau = 1$ yr.
– Relative strength index β is computed w.r.t. $S\&P500$.

*__International Advisory Services Group__ http://iasg.pertrac2000.com/mainframe.asp

Discussion

1. Studied MDD for a Brownian motion.
   – geometric Brownian?

2. We now have scaling laws for MDD and Sterling-type ratios.

3. Portfolio opt. to maximize Calmar?

   Since Clmr is monotonic in Shrp, optimization of Shrp implies optimization of Clmr.

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