The Maximum Drawdown of the Brownian Motion

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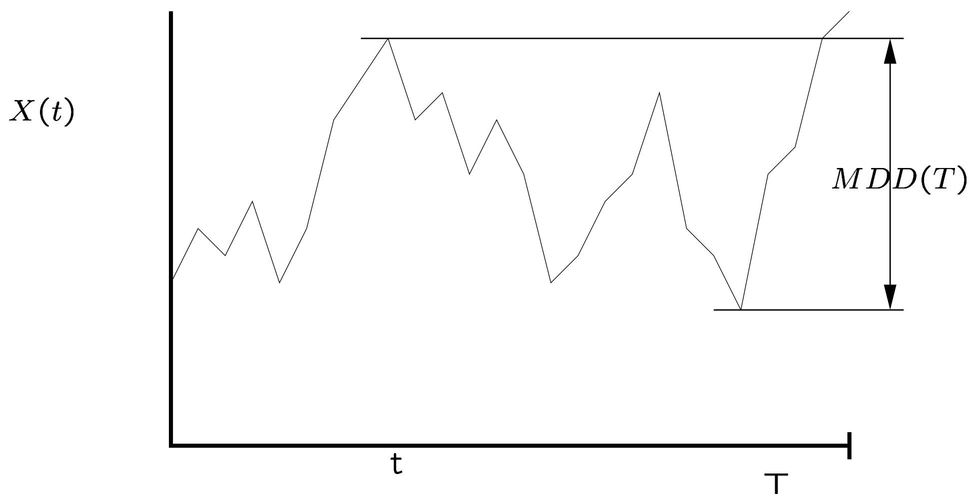
The Maximum Drawdown

The MDD is defined as the maximum loss incurred from peak to bottom during a specified period of time.



$$MDD(T) = \text{sup}_{t\in[0,T]}[\text{sup}_{s\in[0,t]}X(s) - X(t)],$$

where $X(t)$ represents the equity curve of the trading system or fund.

The Sterling Ratio

$$\text{Sterling}(T) = \frac{\text{Return over } [0, T]}{MDD \text{ over } [0, T]}$$

How to scale the sterling ratio?

Sharpe ratio: well known $\sqrt{T}$ rule.

Setup

Trading system is a Brownian motion:

$$dX(t) = \mu dt + \sigma dw(t) \qquad\qquad 0 \leq t \leq T$$

$\mu =$ average return per unit time,
(drift)
$\sigma =$ std. dev. of the returns per unit time,
(volatility)
$dw(t) =$ Wiener increment.

We would like to study $MDD(T)$.

Expected MDD(T)

Theorem:

$$E(MDD) = \begin{cases} \frac{\sigma^2}{\mu} Q_p\left(\frac{\mu^2 T}{\sigma^2}\right) & \text{if } \mu > 0 \\ \sqrt{\frac{\pi}{8}}\sigma\sqrt{T} & \text{if } \mu = 0 \\ \frac{\sigma^2}{\mu} Q_n\left(\frac{\mu^2 T}{\sigma^2}\right) & \text{if } \mu < 0 \end{cases}$$

Q_p and Q_n are **“universal” functions**.
Only need to be computed once!

The Universal Q_p and Q_n

$$Q_p(x) = 2\int_0^\infty du \left[e^{-u} \sum_{n=1}^{\infty} \frac{\sin^3(\theta_n)\left(1-e^{-\frac{x}{2\cos^2(\theta_n)}}\right)}{\theta_n - \cos(\theta_n)\sin(\theta_n)} \right.$$

$$\left. + e^{-\frac{u}{\tanh(u)}} \sinh(u) \left(1 - e^{-\frac{x}{2\cosh^2(u)}}\right) \right]$$

where θ_n are eigen solutions:
$\tan(\theta_n) = \frac{\theta_n}{u}, \ \theta_n \in \left((n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi\right)$

$$Q_n(x) = -2\int_0^\infty du \ e^{u} \sum_{n=1}^{\infty} \frac{\sin^3(\theta_n)\left(1-e^{-\frac{x}{2\cos^2(\theta_n)}}\right)}{\theta_n - \cos(\theta_n)\sin(\theta_n)}$$

where θ_n are eigen solutions:
$\tan(\theta_n) = -\frac{\theta_n}{u}, \ \theta_n \in \left((n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi\right)$

Derivation Trick

Drawdown at t as a stochastic process.

$$X(t)\downarrow \;\Rightarrow DD(t)\uparrow$$
$$X(t)\uparrow \;\Rightarrow DD(t)\downarrow$$

except $DD(t)$ is reflecting at 0.

DD(t) is a **reflected Brownian motion**.

Distribution of extrema of reflected Brownian motion has been studied!

Sharpe vs. MDD

For the case $\mu > 0$:

$$E(MDD(T)) = \frac{\sigma}{\text{Sharpe}} Q_p\left((\text{Sharpe})^2 T\right)$$

Asymptotic Behavior?

Most trading desks are interested in long term performance results.

Asymptotic Behavior

$T \longrightarrow \infty$ (T is the investment horizon).

Theorem:

$$E(MDD) \longrightarrow \begin{cases} \frac{\sigma^2}{\mu}\left(-0.54 + \frac{1}{2}\log(T) + \log\frac{\mu}{\sigma}\right) \\ \\ \sqrt{\frac{\pi}{8}}\sigma\sqrt{T} \\ \\ \mu T \end{cases}$$

Three different types of behavior:

$$\log(T),\ \sqrt{T},\ T$$

depending on the regime of μ.

Asymptotic Sharpe vs. Sterling

Focusing again on $\mu > 0$,

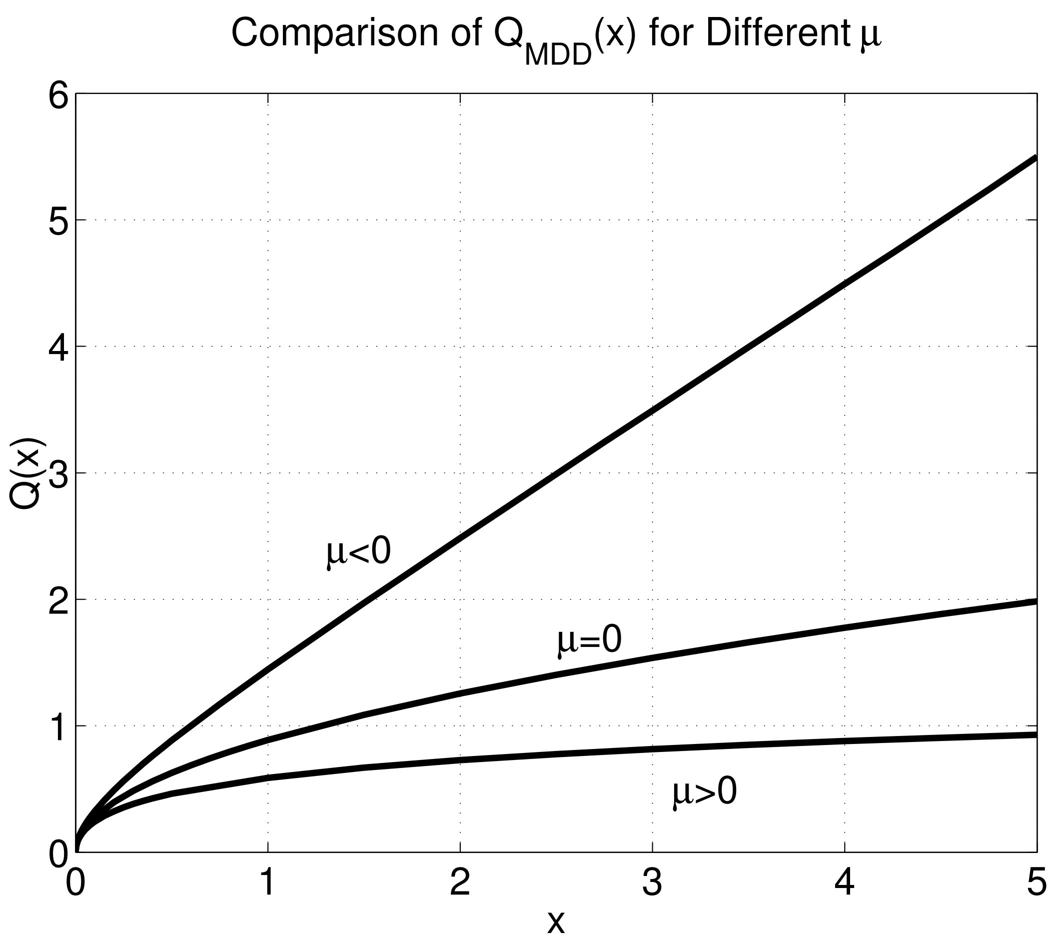
$$\text{Sterling}(T) = \frac{(\text{Sharpe})^2}{Q_p\left((\text{Sharpe})^2 T\right)}$$

$T \longrightarrow \infty$,

$$\text{Sterling}(T) \longrightarrow \frac{(\text{Sharpe})^2}{-0.54 + \frac{1}{2}\log(T) + \log(\text{Sharpe})}$$

Note: Sharpe fixed$\Rightarrow$ scale Sterling by $\log(T)$ to get a meaningful statistic.

Behavior of $Q_{MDD}(x)$



Discussion

1. Studied MDD for a Brownian motion.
   – geometric Brownian?

2. We now have the asymptotics of MDD.
   – can meaningfully scale Sterling.

3. Portfolio opt. to maximize Sterling?

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Thank You!