Discrete Energy Minimization with Graph Cuts

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Problem

Energy Minimization of discrete, non-convex functions

- Comes up frequently in "Early Vision" problems
- Other problems can be formulated as energy minimization
- In some cases it performs much better than typical optimization routines

Energy Functions of F^2

Energy functions in the class F^2 can be written as

- For a function $E(x_1, ...x_n)$ of n variables s.t

$$x_1, ...x_n, x_i \in L$$

- A labeling f assigns a value to each $x_i \in L$. The energy of the labeling f is given by

$$E(f) = \sum_i D_i(f_i) + \sum_{i<j} V_{i,j}(f_i, f_j)$$

Other Classes

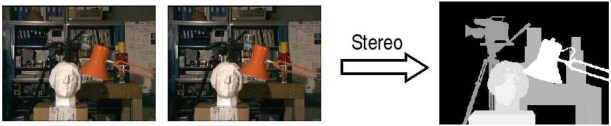
Energy functions in the class F^3 can be written as

$$E(f) = \sum_i D_i(f_i) + \sum_{i<j} V_{i,j}(f_i, f_j) + \sum_{i<j<k} V_{i,j,k}(f_i, f_j, f_k)$$

Class F^1 can be written as

$$E(f) = \sum_i D_i(f_i)$$

Example: Stereo Vision



Let p and q be pixels in the first image of a stereo pair. Let the function f map pixels in the first image to pixels in the second image. The energy function is

$$E(f) = \sum_{p} D_p(f_p) + \sum_{p,q \in N} V_{p,q}(f_p, f_q)$$

Bayesian Model

The Energy function is created by treating the image as a Markov random field. The inference rule is that pixels that are close together are usually similar.

Min-Cut formulation: Binary Labels

- Let the set of possible labels $L = 0, 1$
- Any labeling function partitions the variables into two sets 0,1 and the goal is to find the labeling s.t. the energy is minimum.
- This partitioning problem can be solved optimally by the max-flow, min-cut algorithm.

Graph Construction and Submodularity

What is the criteria for a binary subproblem to be graph representable/minimized? For a binary function

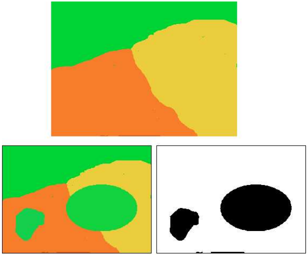
$$E(x_1, ...x_n) = \sum_i E^i(x_i) + \sum_{i<j} E^{i,j}(x_i, x_j)$$

The function is submodular and therefore graph representable if it satisfies

$$E^{i,j}(0,0) + E^{i,j}(1,1) \leq E^{i,j}(0,1) + E^{i,j}(1,0)$$

Partitioning Algorithms: α Expansion

- For some label α, and some current labeling f, determine the minimum energy according to the following rule:
- For all pixels p that are not labeled with α, either p keeps its label, or it switches its label to α.



Expansion move   Binary image

Partitioning Algorithms: $\alpha - \beta$ Swap

For some pair of labels α and β, and some current labeling f, determine the labeling with the minimum energy according to the following rule:
For all pixels p that are labeled with α or β, label p with either α or β.

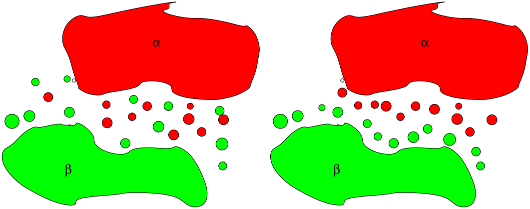
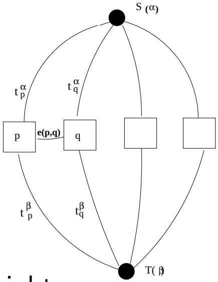


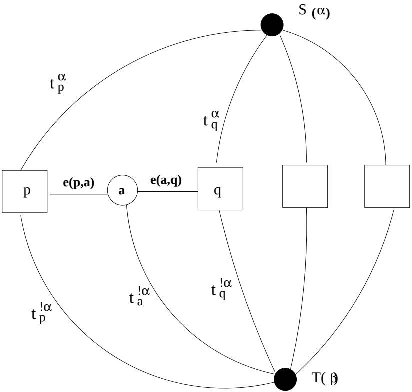
Figure: Before and After

$\alpha - \beta$ Swap as a Min-cut problem



edge	weight	for
t_p^α	$D_p(\alpha) + \sum_{q \in N_p, q \notin P_{\alpha,\beta}} V_{p,q}(\alpha, f_q)$	$p \in P_{\alpha\beta}$
t_p^β	$D_p(\beta) + \sum_{q \in N_p, q \notin P_{\alpha,\beta}} V_{p,q}(\beta, f_q)$	$p \in P_{\alpha\beta}$
$e_{p,q}$	$V_{p,q}(\alpha, \beta)$	$p, q \in N, p, q \in P_{\alpha\beta}$

α-Expansion as a Min-cut problem



α-Expansion (cont.)

edge	weight	for
$t_p^{!\alpha}$	∞	$p \in P_\alpha$
$t_p^{!\alpha}$	$D_p(f_p)$	$p \notin P_\alpha$
t_p^{α}	$D_p(\alpha)$	$p \in P$
$e_{p,a}$	$V_{p,q}(f_p, \alpha)$	$p, q \in N, f_p \neq f_q$
$e_{a,q}$	$V_{p,q}(\alpha, f_q)$	$p, q \in N, f_p \neq f_q$
$t_a^{!\alpha}$	$V_{p,q}(f_p, f_q)$	$p, q \in N, f_p \neq f_q$
$e_{p,q}$	$V_{p,q}(f_p, \alpha)$	$(p, q) \in N, f_p = f_q$

The Target Assignment Problem

Problem: Given m targets in the plane, and n stereo sensors, determine the placement of the sensors that minimizes the uncertainty given by

$$\frac{d(s_1, t)\, d(s_2, t)}{\sin \angle s_1 t s_2}$$

where s_1 and s_2 are sensors and tracking the target t

The Target Assignment Problem (cont.)

As an energy minimization problem:

- Discretize the possible placements of the sensors
- The variables are the sensors and the labels are the set of possible locations.
- The smoothness cost is zero if the sensors are not tracking a common target, otherwise it is

$$\frac{d\left(s_1, t\right) d\left(s_2, t\right)}{\sin \angle s_1 t s_2}$$

- For a single iteration, the set of possible labels for each sensor is restricted to a local grid.

References

- V. Kolmogorov and R. Zabih. "What energy functions can be minimized via graph cuts? " *In European Conference on Computer Vision*, 2002
- Yuri Boykov, Olga Veksler, Ramin Zabih. "Fast Approximate Energy Minimization" *In IEEE transactions on Pattern Analysis and Machine Intelligence* (PAMI), vol. 23, no. 11, pp. 1222-1239, 2001