Announcements

- Rainbow grades
  - Quizzes 1-4, Homework 1-2
  - Exam 1
    - 16% A-range, 27% B-range, 27% C-range, 16% D-r
    - Median: 91/115, Mean: 88/115
    - We will release papers on Submitty on Thu. Thanks!
- Drop deadline is this Friday, October 21st!
- We'll start grading HW3 next week. Thanks!

Functional Programming with Scheme

Keep reading: Scott, Chapter 11.5-11.6

Lecture Outline

- Notes on writing “comments” for homework
- Scheme
  - Equality testing
  - Higher-order functions

    reduce
  - `map, foldr, foldl`
  - Tail recursion

Writing “Comments”

Each function should have the following sections:

```
;; Contract: len : (list a) -> integer
;; Purpose: to compute length of a list lis
;; Example: (len '(1 2 3 4)) should return 4
;; Definition:
(define (len lis)
   (if (null? lis) 0 (+ 1 (len (cdr lis)))))
```

Writing “Comments”

;; Contract:

;; **len** : (list a) -> number

- Has two parts. The first part, to the left of the colon, is the name of the function. The second part, to the right of the colon, states what type of data it consumes and what type of data it produces
- We shall use a, b, c, etc. to denote type parameters, and list to denote the list type
- Thus, **len** is a function, which consumes a list whose elements are of some type a, and produces a number

Writing “Comments”

;; Contract:

;; **lis** : (list integer) -> (list integer)

- **lis** : is a function, which consumes a list of integers, and produces a list of integers (you may assume that the input is a list of integers)

;; Purpose: to compute …

sublists : integer * (list a) → (list (list a))

integer, (list a) → ...

Writing “Comments”

- Comments are extremely important in Scheme

	Scheme	Java	C++	Dafny
Type signature	✗	✓	✓	✓
Behavioral spec	✗	✗	✗ !	✓

- Why?
- Our “comments” amount to adding an <u>unchecked</u> type signature + an informal behavioral specification

Lecture Outline

- Notes on writing “comments” for homework
- Scheme
  - Equality testing
  - Higher-order functions
  - **`map, foldr, foldl`**
  - Tail recursion

Equality Testing

eq?

- Built-in predicate that can check atoms for equal values
- Does not work on lists in the way you might expect!

eql?

- Our predicate that works on lists

```
(define (eql?  x  y)
  (or (and (atom?  x) (atom? y) (eq?  x  y))
      (and  (not (atom? x)) (not (atom? y))
            (eql? (car x) (car y))
            (eql? (cdr x) (cdr y)) )))
```

equal?

- Built-in predicate that works on lists

Examples

`(eql? '(a) '(a))` **yields** what?

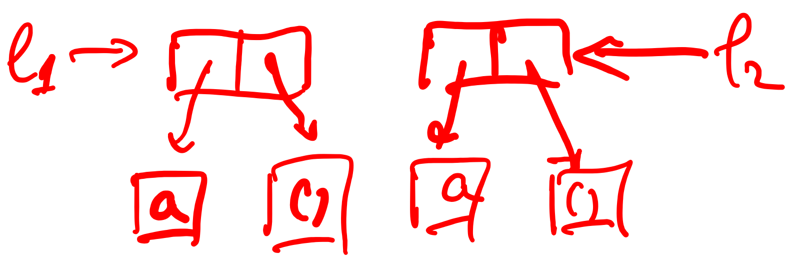
`(eql? 'a 'b)` **yields** what?

`(eql? 'b 'b)` **yields** what?

`(eql? '((a)) '(a))` **yields** what?

`(eq? 'a 'a)` **yields** what?

`(eq? '(a) '(a))` **yields** what?



Models for Variables

- Value model for variables
  - A variable is a location that holds a value
    - I.e., a named container for a value
  - a := b

l-value (the location)

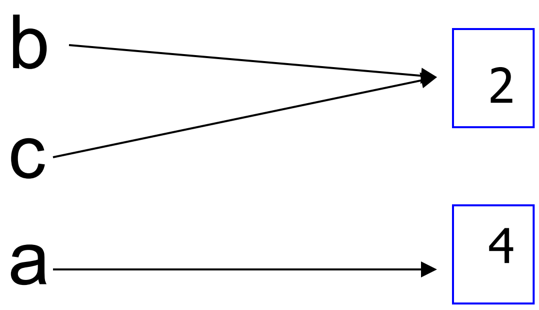
r-value (the value held in that location)

- Reference model for variables
  - A variable is a reference to a value
  - Every variable is an l-value
    - Requires dereference when r-value needed (usually, but not always implicit)

Models for Variables: Example

```
b := 2;
c := b;
a := b + c;
```

- Value model for variables
  - b := 2 b: 2
  - c := b c: 2
  - a := b+c a: 4
- Reference model for variables
  - b := 2
  - c := b
  - a := b+c



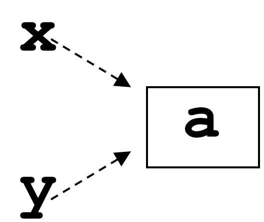
b =?= c

Equality Testing: How does eq? work?

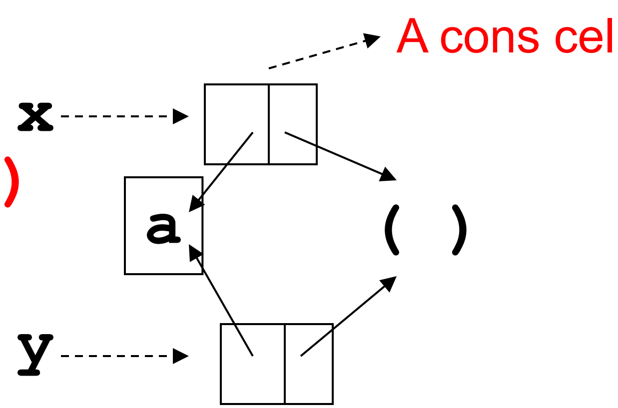
- Scheme uses the reference model for variables!

```
(define (f x y) (list x y))
```

Call `(f 'a 'a)` yields `(a a)`
`x` refers to atom `a` and `y` refers to atom `a`.
`eq?` checks that `x` and `y` both point to the
same place.



Call `(f '(a) '(a))` yields `((a) (a))`
`x` and `y` do not refer to the same list.



Models for Variables

- C/C++, Pascal, Fortran
  - Value model
- Java
  - Mixed model: value model for simple types, reference model for class types
- JS, Python, R, etc.
  - Reference model
- Scheme
  - Reference model! **`eq?`** is “reference equality” (akin of Java’s ==), **`equal?`** is value equality

Equality Testing

- In languages with reference model for variables we have two tests for equality
  - One tests reference equality, whether two references refer to the same object
    - **`eq?`** in Scheme **`(eq? '(a) '(a))`** yields **`#f`**
    - **`==`** in Java   is in Python
  - Other tests value equality. Even if the two references do not refer to the same object, they may refer to objects that have the same value
    - **`equal?`** in Scheme **`(equal? '(a) '(a))`** yields **`#t`**
    - **`.equals()`** method in Java

Lecture Outline

- Notes on writing “comments” for homework
- Scheme
  - Equality testing
  - Higher-order functions
  - **`map, foldr, foldl`**
  - Tail recursion

Higher-order Functions

- Functions are first-class values

- A function is said to be a higher-order function if it takes a function as an argument or returns a function as a result

- Functions as arguments

  ```
  (define (f g x) (g x))
  ```

  `(f number? 0)` **yields** `#t`

  `(f len '(1 (2 3)))` **yields** what? 2

  `(f (lambda (x) (* 2 x)) 3)` **yields** what? 6

Higher-order Functions

- Functions as return values

```
(define (fun)
    (lambda (a) (+ 1 a)))
```

`(fun 4)` yields what? ERROR

`((fun) 4)` yields what?

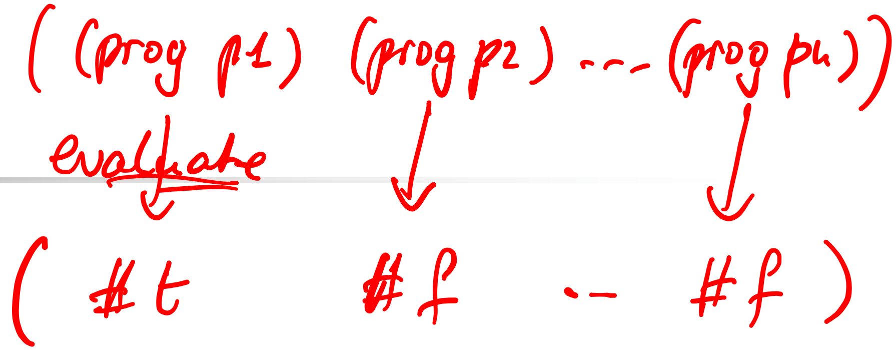
λa. a+1

Higher-order Functions: **`map`**

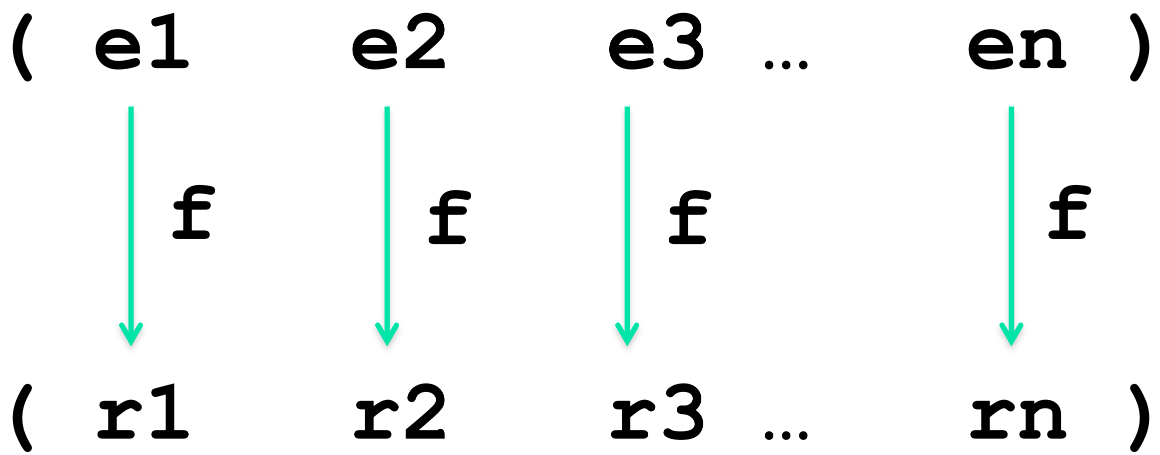
- Higher-order function used to apply another function to every element of a list
- Takes 2 arguments: a function **`f`** and a list **`lis`** and builds a new list by applying the **`f`** to each element of **`lis`**

```
(define (my-map f lis)
  (if (null? lis) '()
    (cons (f (car lis)) (my-map f (cdr lis)))))
```

map



(my-map f lis)



There is a build-in function **`map`**

map

```
(define (my-map f l)
    (if (null? l) '( )
      (cons (f (car l)) (my-map f (cdr l)) )))
```

`(my-map abs '(-1 2 -3 -4))` yields `(1 2 3 4)`

`(my-map (lambda (x) (+ 1 x)) '(1 2 3))` yields what?

(2 3 4)

`(my-map (lambda (x) (abs x)) '(-1 2 -3))` yields what?

(1 2 3)

map

(apply + (1 2 3)) → 6

Remember **atomcount**, calculates the number of atoms in a list

```
(define (atomcount s)
   (cond ((null? s) 0)
         ((atom? s) 1)
         (else (+ (atomcount (car s))
                  (atomcount (cdr s))))))
```

We can write **atomcount2**, using map:

```
(define (atomcount2 s)
   (cond ((atom? s) 1)
         (else (apply + (map atomcount2 s)))))
```

map

```
(define (atomcount2 s)
  (cond ((atom? s) 1)
    (else (apply + (map atomcount2 s)))))
```

`(atomcount2 '(1 2 3))` **yields** 3

`(atomcount2 '((a b) d))` **yields** 3

`(atomcount2 '(1 ((2) 3) (((3) (2) 1))))` ?

Question

(atom? '()) → #t

My **atomcount2** defined below

```
(define (atomcount2 s)
   (cond ((atom? s) 1)
         (else (apply + (map atomcount2 s)))))
```

((null? s) 0)

has a bug :). Can you find it?

Answer: It counts the null list `'()` as an atom.
E.g., `(atomcount2 '())` will return 1.

Exercise

```
(define (atomcount2 s)
  (cond ((atom? s) 1)
        (else (apply + (map atomcount2 s)))))
```

Now, let's write **flatten2** using **map**

`(flatten2 '(1 ((2) 3) (((3) (2) 1))))` **yields**
`(1 2 3 3 2 1)`

```
(define (flatten2 s)
    …
```

Hint: you can use **(apply append (…**

Exercise

```
(flatten '(1 (2 (3 (c))))) → (1 2 3 c)
(apply append ((a) (2 3 c))) → (a 2 3 c)

(define (flatten s)
  (cond ((null? s) '())
        ((atom? s) (list s))
        (else (append (flatten (car s))
                      (flatten (cdr s))))))
                      ; (flatten (car s)) → (1 2 3)

(list 'a) → (a)

(define (flatten2 s)
  (cond ((null? s) '())
        ((atom? s) (list s))
        (else (apply append (map flatten2 s)))
  )
)
```

foldr

- Higher-order function that "folds" ("reduces") the elements of a list into one, from right-to-left
- Takes 3 arguments: a binary operation op, a list lis, and initial value id. forldr "folds" lis

```
(define (foldr op lis id)
     (if (null? lis)  id
          (op (car lis) (foldr op (cdr lis) id))  ))
```

(foldr +   '(10 20 30) 0) **yields** 60

it is 10 + (20 + (30 + 0))

(foldr - '(10 20 30) 0) **yields** ?

foldr

(foldr op lis id)

(e_1 … e_{n-1} op e_n) op id

(e_1 … e_{n-1}) res_1

…

(e_1) op res_{n-1}

<u>res_n</u>

Exercise

- What does
  (foldr append ‘((1 2) (3 4)) ‘()) yield?
  Recall that append appends two lists:
  (append ‘(1 2) ‘((3) (4 5))) yields (1 2 (3) (4 5))

- Now, define a function len2 that computes the length of a list using foldr
  (define (len2 lis)
  	(foldr …

foldr

op

List element

(define (len2 lis) (foldr (lambda (x y) (+ 1 y)) lis 0))

Partial result

(a b c) 0

(a b) 1

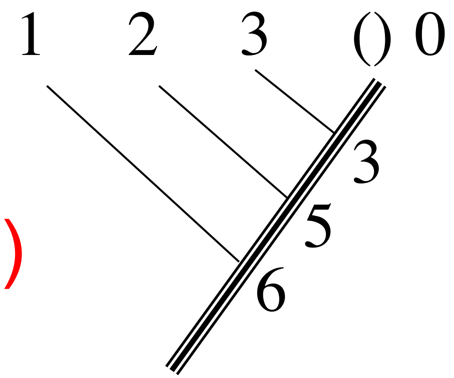
(a) 2

3

foldr

- foldr is right-associative
  - E.g., (foldr + ‘(1 2 3) 0) is 1 + (2 + (3 + 0))
  - Partial results are calculated in order down the else-branch

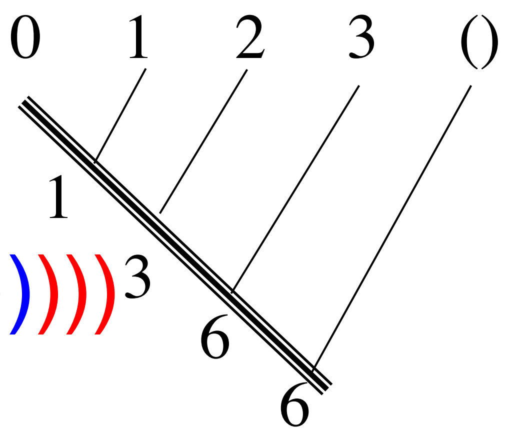
```
(define (foldr op lis id)
   (if (null? lis)  id
      (op (car lis) (foldr op (cdr lis) id))))
```



foldl

- foldl is left-associative and (as we shall see) more efficient than foldr
  - E.g., (foldl + ‘(1 2 3) 0) is ((0 + 1) + 2) + 3
  - Partial results are accumulated in id

```
(define (foldl op lis id)
  (if (null? lis) id
    (foldl op (cdr lis) (op id (car lis)))))
```



foldl

(foldl op lis id)

$id \; (\; e_1 \; e_2 \; e_3 \; \ldots \; e_n \;)$

$id_1 \; (\; e_2 \; e_3 \; \ldots \; e_n \;)$

$id_2 \; (\; e_3 \; \ldots \; e_n \;)$

$\ldots$

$id_{n-1} \; (\; e_n \;)$

id_n

Exercise

```
(define (foldl op lis id)
    (if (null? lis) id
        (foldl op (cdr lis) (op id (car lis)))))
```

- Define a function rev computing the reverse of a list using foldl
  E.g., (rev ‘(a b c)) yields (c b a)

```
(define (rev lis)
    (foldl (lambda (x y) (…
```

foldl

op id

Next element

(define (rev lis) (foldl (lambda (x y) (cons y x)) lis ‘()))

Partial result

() (a b c)

(a) (b c)

(b a) (c)

(c b a) ()

Exercise

```
(define (foldr op lis id)
   (if (null? lis)  id
      (op (car lis) (foldr op (cdr lis) id))  ))
(define (foldl op lis id)
   (if (null? lis) id
      (foldl op (cdr lis) (op id (car lis)))) )
```

- Write len, which computes the length of the list, using foldl
- Write rev, which reverses the list, using foldr

Exercise

```
(define (foldl op lis id)
  (if (null? lis) id (foldl op (cdr lis) (op id (car lis)))) )
```

- Write len, which computes the length of the list, using foldl

```
(define (len lis) …
```

Exercise

Can you write the contract for foldl ?

```
(define (foldl op lis id)
    (if (null? lis) id
        (foldl op (cdr lis) (op id (car lis)))))
```

;; Contract: (b*a->b) * (list a) * b -> b

;; foldl : (b * a -> b) * (list a) * b -> b

Exercise

How about the contract for foldr ?

```
(define (foldr op lis id)
    (if (null? lis) id
        (op (car lis) (foldr op (cdr lis) id))))
```

```
;; Contract:
;; foldr : (a * b -> b ) * (list a) * b  ->  b
```

foldr vs. foldl

```
(define (foldr op lis id)
   (if (null? lis)  id
      (op (car lis) (foldr op (cdr lis) id))  ))
(define (foldl op lis id)
   (if (null? lis) id
      (foldl op (cdr lis) (op id (car lis)))  ))
```

- Compare underlined portions of these 2 functions
  - foldr contains a recursive call, but it is not the entire return value of the function
  - foldl returns the value obtained from the recursive call to itself!

Tail Recursion

- If the result of a function is computed without a recursive call OR it is the result of an <u>immediate</u> recursive call, then the function is said to be tail recursive
  - E.g., foldl
- Tail recursion can be implemented efficiently
  - Result is accumulated in one of the arguments, and stack frame creation can be avoided!
  - Scheme implementations are required to be “properly tail-recursive”

Tail Recursion: Two Definitions of Length

```
(define (len lis)
  (if (null? lis)
      0
      (+ 1 (len (cdr lis)))))

(len '(3 4 5))
```

```
(define (lenh lis total)
  (if (null? lis)
      total
      (lenh (cdr lis) (+ 1 total))))

(define (len lis) (lenh lis 0))

(len '(3 4 5))
```

Lenh is tail recursive. total accumulates the length

Tail Recursion: Two Definitions of Factorial

```
(define (factorial n)
        (cond ((zero? n) 1)
              ((eq? n 1) 1)
        (else  (* n (factorial (- n 1))))))
```

```
(define (fact2 n acc)
  (cond ((zero? n) 1)
        ((eq? n 1) acc)
    (else (fact2 (- n 1)
          (* n acc))))))
```

```
(define (factorial n)
    (fact2 n 1))
```

fact2 is tail recursive

The End