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NOTES

How to Spread Rumors Fast

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The problem Seven gossiping friends are most anxious to share their gossip amongst each other. Each gossiper knows something unique to begin with. Each day, some of the gossipers phone each other and exchange all the gossip they have collected so far, but on any given day, each gossiper can be involved in a phone call with at most one other person. (No conference calls allowed!)

What is the minimum number of days it would take for each gossiper to learn the collective gossip of all the gossipers?

Let us denote by $d(n)$ the minimal number of days required for n gossipers to share all their knowledge. The first few values of $d(n)$ are as follows:

n	1	2	3	4	5	6	7	8
$d(n)$	0	1	3	2	4	3	4	3

Note that $d(n)$ is not an increasing function: for instance, it takes more time for three people to share their gossip than it does for four people! We shall prove

$$d(n) = \begin{cases} \lceil \log_2 n \rceil, & \text{if } n \text{ is even,} \\ \lceil \log_2 n \rceil + 1, & \text{if } n \text{ is odd,} \end{cases}$$

and indicate how the gossipers should behave to achieve this minimum.

A lower bound On any given day, the number of people who can know a given fact, say, fact A, can at most double since each person who knows fact A can tell it to at most one other person. Therefore, we cannot hope to spread all the gossip around in fewer than $\lceil \log_2 n \rceil$ days, where $\lceil x \rceil$ is the least integer greater than or equal to x.

Hypercubes Consider an even number of gossipers $2m$. On the first day, we can divide the gossipers into two groups, group I and group II, each of size m. By definition, in $d(m)$ days, each member of each group can learn all the information within his or her own group. We take $d(m)$ days to accomplish this, working the two groups in parallel. Next, we pair up the gossipers in group I arbitrarily with the gossipers in group II so that after another day, all the gossipers will know all the gossip. In this way, we see that $d(2m) \le 1 + d(m)$.

When $n = 2^D$, this inductively leads to the inequality $d(2^D) \leq D$. On the other hand, we have already seen that $d(2^D) \geq D$. Consequently, $d(2^D) = D$. In this case, the gossipers can be identified with vertices of a D-dimensional hypercube so that the phone calls made on any given day correspond to the set of edges parallel to some edge.

An odd number of gossipers   Now assume n is odd and greater than 1. Here is one method of spreading the rumors. Let h be the positive integer with $2^h < n < 2^{h+1}$. Let H be a subset of 2^h gossipers, and let H^c be the complement of H. Note that $|H| > |H^c|$. Because of this, on the first day we can pair up the members of H^c with distinct members of H. (Some members of H will not talk to anyone on the first day.) After the first day, the members of H will collectively know everything. We can use the results of the previous section and, ignoring temporarily the members of H^c, have the members of H take h days to exchange their knowledge amongst each other. When that is finished, we again pair up members of H^c with the now omniscient members of H, thus taking another day to complete the spread of rumors. See the illustration for the case $n = 7$ (lines with numbers indicate the day on which the connected couple should communicate). This process takes $h + 2 = \lceil \log_2 n \rceil + 1$ days.

For n odd, it turns out that this is the best one can do. To see why, observe that on the first day of conversations, at least one gossiper will be forced to sit out from communicating. This gossiper knows something unknown to the others (by our hypothesis), and the lower bound argument given earlier shows that this bit of information takes at least $\lceil \log_2 n \rceil + 1$ days to spread around because on the second day, still only one person knows the bit of information. We conclude that for n odd, $d(n) = \lceil \log_2 n \rceil + 1$.

An even number of gossipers   The method used to spread rumors for odd n applies in general. Thus, we see that

$$\lceil \log_2 n \rceil \leq d(n) \leq \lceil \log_2 n \rceil + 1.$$

When n is odd, we have equality on the right, and when $n = 2^d$, we have equality on the left. We now discuss the remaining case, with n even but not a power of 2. (Actually, the method we describe below also works for $n = 2^d$, and in that case gives a new pattern of phone calls that distributes the information just as fast as the hypercube arrangement.)

For $k \geq 1$, define

$$\Delta_k = \frac{1 - (-2)^k}{3}.$$

Note that Δ_k is an odd integer. Label the gossipers 1 through n once and for all. By gossiper p (p any integer), we shall be referring to the gossiper labeled q where $1 \leq q \leq n$ and q and p are congruent modulo n. Since n is even, it makes sense to refer to even and odd gossipers. On day k, make the odd gossiper $2x - 1$ call up the even gossiper $2x - 1 + \Delta_k$.

We claim that at the end of day k, gossipers $2x - 1$ and $2x - 1 + \Delta_k$ each know all the knowledge originally held by gossipers $2x - 1 + \Delta_k/2 - (2^{k-1} - \frac{1}{2})$ through $2x - 1 + \Delta_k/2 + (2^{k-1} - \frac{1}{2})$. Note that when $k \leq \log_2 n$, this implies that each gossiper knows 2^k pieces of information at the end of day k, the maximum possible. And at the end of day $\lceil \log_2 n \rceil$, each gossiper knows everything.

We prove the claim by induction on k. When $k = 0$, we have $\Delta_k = 0$ and the claim is that gossiper $2x - 1$ has the knowledge of gossiper $2x - 1$, which is clear. Assume the claim is true for $k = K$, where K is a nonnegative integer. On day $K + 1$, gossiper $2x - 1$ and gossiper $2x - 1 + \Delta_{K+1}$ exchange information. By the inductive hypothesis, gossiper $2x - 1$ has the original knowledge of gossipers l_1 through r_1, and gossiper $2x - 1 + \Delta_{K+1}$ has the original knowledge of gossipers l_2 through r_2, where

$$l_1 = 2x - 1 + \frac{\Delta_K}{2} - \left(2^{K-1} - \frac{1}{2}\right),$$

$$r_1 = 2x - 1 + \frac{\Delta_K}{2} + \left(2^{K-1} - \frac{1}{2}\right),$$

$$l_2 = 2x - 1 + \Delta_{K+1} - \frac{\Delta_K}{2} - \left(2^{K-1} - \frac{1}{2}\right),$$

$$r_2 = 2x - 1 + \Delta_{K+1} - \frac{\Delta_K}{2} + \left(2^{K-1} - \frac{1}{2}\right).$$

A straightforward computation shows that $r_1 + 1 = l_2$ if K is even, and $r_2 + 1 = l_1$ if K is odd (in other words the two blocks of integers in $[l_1, r_1]$ and $[l_2, r_2]$ are contiguous), and that in either case, after day $K + 1$, gossipers $2x - 1$ and $2x - 1 + \Delta_{K+1}$ each know the original knowledge of gossipers $2x - 1 + \Delta_{K+1}/2 - (2^K - \frac{1}{2})$ through $2x - 1 + \Delta_{K+1}/2 + (2^K - \frac{1}{2})$. The claim follows by induction.

Thus, using this strategy, each gossiper will know everything at the end of day $\lceil \log_2 n \rceil$.

Final remarks Admittedly, one day is a bit much for the time it takes for even the most loquacious of rumormongers to communicate with a friend! Perhaps one hour or even one minute would have been more realistic.

Lest the reader find the topic of this note too frivolous, let us remark that the problem can also be interpreted in terms of communications between computers operating in parallel. The result can be useful, for instance, when multiple copies of a database are distributed to several locations and replication is needed to keep the databases synchronized. In fact, this is the situation in which our problem originally arose.