Subh: A game on digraphs[*]

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1 Introduction

There are many interesting games on graphs [**?**]. This paper describes a new game which incorporates a new idea. In traditional games on graphs, the two players alternately choose edges such that the first player wants to attain a property and the second player wants to attain a similar property. For example in Sim [**?**], the two players alternately color an edge (in a complete graph on six nodes) with either red or blue. (The first player chooses the color red and the second player chooses the color blue.) The first player's aim is to get a red triangle and the second player's aim is to get a blue triangle. (By a triangle, we mean a cycle of length three.) The person who gets a triangle of his color wins. The game presented in this paper is different from the traditional ones in the sense that one player pursues a property while the other player tries to block it.

Our proposed game is as follows: Let us play on a five node directed graph. Initially there are no edges in the directed graph. The players alternately add a directed edge between any pair of nodes, say u and v, provided there is no edge between u and v already, and there is no edge between v and u. The aim of the first player is to create a directed Hamiltonian cycle (i.e., a directed cycle that passes through all the nodes of the graph once and only once). The aim of the second player is to prevent it. The game ends when the first player creates a Hamiltonian cycle and, hence, the first player wins or no more directed edges can be added and, therefore, the second player wins. (Subh is an abbreviation for Subgraph which is Hamiltonian.)
The first player could use some or all of the directed edges that the second player has added to form a Hamiltonian cycle. The reader is urged to try this game before proceeding further. We use standard definitions in Graph Theory [**?**]. Further, we can play this game on a directed graph with any number of nodes with the same winning conditions. Also, the roles of player could be reversed, the first player's aim is to block a Hamiltonian cycle, while the second player's aim is to create a Hamiltonian cycle.

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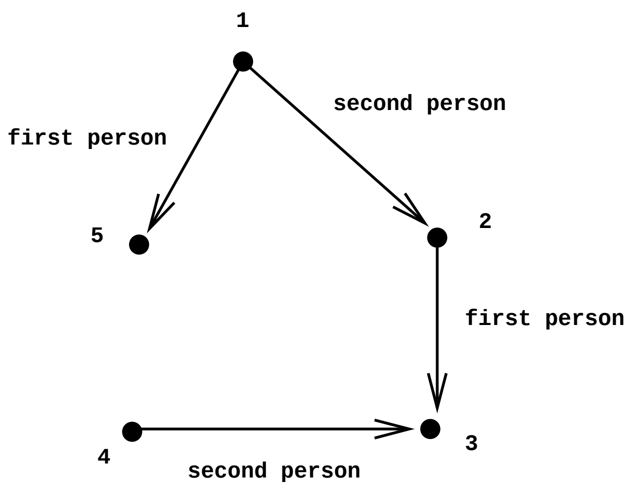

Figure 1: The first two moves of a game.

2 Example

Let us illustrate this game. The first person could draw an edge between any two nodes. Let the first person draw an edge from 1 to 5. Now that the edge between 1 and 5 is taken, the second player can draw any of the remaining nine edges. Let the second person draw an edge from node 1 to node 2. The first person draws an edge from node 2 to node 3. (Note that the first person who wants to create a Hamiltonian cycle could use the edge that was drawn by the second person.) As a response to this, the first person draws an edge from node 4 to node 3. The state of the graph after this move is shown in Figure 1. Figure 2 describes the state of a graph for some other game. If the first player does not draw an edge from node 2 to node 5, he will lose, as the second player will draw an edge from node 5 to node 2 and there can be no Hamiltonian cycle in the graph. Figure 3 describes the state of a graph for yet another game.

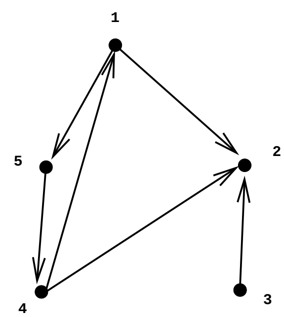

Figure 2: The first player's turn.

Here, the second player draws an edge from node 3 to node 5 and he wins as there can be no Hamiltonian cycle in the graph.

3 Strategy

We will try to obtain a strategy for the first player on local properties of the graph in which the game is played. A *tournament* is a directed graph, whose underlying undirected graph is a complete graph, i.e., a directed edge exists between every unordered pair of nodes. One could conjecture that if every node has nonzero indegree and nonzero outdegree, then it has a Hamiltonian cycle. The conjecture is true for n=3, 4 and 5. For $n \geq 6$ the conjecture is false. The proof for n =3 and 4 are obvious and for $n = 5$, we give the proof in Lemma 1.

Lemma 1 : In a directed graph (whose underlying undirected graph is a complete graph of five nodes), if the indegree and outdegree of each node is greater than 0, then there exists a directed Hamiltonian cycle.

Proof: Suppose the maximum length of the directed cycle is three. (Recall that there can be no cycle of length two.) Without loss of generality, let v_1, v_2, v_3 be the nodes in this cycle, and let the edge between v_4 and v_5 be directed from v_4 to v_5. We shall show that there exist two edges, the one from v_i to v_4 and the other from v_5 to a node v_j, for a pair of distinct nodes

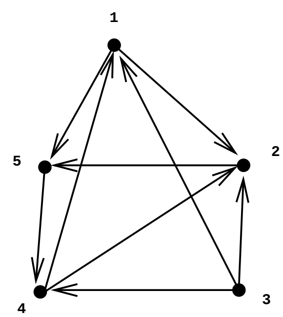

Figure 3: The second player's turn.

v_i and v_j in the cycle, which implies a cycle of length five.

All three edges linking the three nodes of the cycle and v_4 should be directed to v_4. Otherwise, either the indegree of v_4 is zero (if all the edges are directed to the cycle), or the graph has a cycle of length four, contradicting our assumption. Since we cannot have an outgoing edge from v_5 to v_4, there should be an edge from v_5 to a node in the cycle. Let this node be v_j, $1 \leq j \leq 3$. Since all the three edges linking the cycle and v_4 are directed to v_4, there must be an edge going from a node v_i, $i \neq j$, in the cycle to v_4. We have a cycle of length five, a contradiction.

Now, suppose the maximum length of the directed cycle is four. Assume that v_1, v_2, v_3, v_4 are on the cycle in that order. There should be a pair of edges from v_i, $1 \leq i \leq 4$, to v_5 and from v_5 to v_j, $j = (i \bmod 4) + 1$. Otherwise, either the indegree or the outdegree of v_5 should be zero, violating the condition of the lemma. It follows that the graph has a cycle of length five, a contradiction.

We want to reiterate that the above is not valid for $n \geq 6$. Note that the directed graph that results at the end of the game, if played until no more edges could be added, is a tournament [**?**]. Further the above lemma can be derived as a corollary of a stronger result due to Moon [**?**]. We state and prove the lemma in this fashion as it is simple. Moreover, the above lemma yields a strategy for both players. The first player's aim at each turn is to maximize the number of nodes with both indegree and outdegree > 0. The aim of the second player is to maximize the number of nodes with indegree or outdegree $= 0$. When the number of nodes is three or four, the second player always wins, provided he uses the above strategy. However, when n is five, the first player always wins as there are enough edges to make the first player follow the optimal strategy. The first player uses the following strategy in order of applicability:

1. If there is a node with indegree = 3 or a node with outdegree = 3, then add an edge going out or coming into that node.

2. If there are two nodes such that for one the outdegree is 0 and indegree is > 0, and the other one whose indegree is 0 and outdegree is > 0 and there is no edge between them, then add a directed edge from the node with outdegree = 0 to a node with indegree = 0.

3. Find a node with maximum indegree (outdegree is 0) and a node with maximum outdegree (with indegree is 0). If the maximum indegree is higher than the maximum outdegree, then add a directed edge from that node to some other node. If the maximum outdegree is higher, then add a directed edge to that node from some other node. In case both are maximum or the same, choose either of the above.

Note that the strategy for both players involves only local information and is not concerned with the Hamiltonian cycle, which is a global property. Hence, both of them could be efficiently implementable by a fast algorithm.

This game could easily be extended to graphs with more than five nodes. It is important to note that conjecture about nonzero indegree and nonzero outdegree implies the existence of a Hamiltonian cycle is false for $n > 5$, as we can construct two strongly connected components having $n \geq 3$ nodes and can direct edges from nodes of one component to nodes of the other component. However, using the strategy for five nodes, we claim that the first player has a winning strategy.
For example, when the number of nodes is six, the following strategy can be adopted by the first player. The first player designates v_6 as a special vertex. The strategy is to play the strategy stated for five nodes, when the second player adds a directed edge in those five nodes. Whenever the second player adds a directed edge from one of v_2, v_3, v_4 and v_5 to v_6, say v_3 and v_6, the first player adds a directed edge from some other node other than v_3 and v_6; say, between v_4 and v_6, but in the opposite direction. This strategy will always yield a Hamiltonian cycle for the first player. This can be seen as follows: The strategy in the five nodes will yield a cycle of length five in those five nodes. Let that cycle be v_a, v_b, v_c, v_d, and v_e. Suppose there is a directed edge from v_a to v_6. Then if there is a directed edge from v_6 to v_b, we are finished. Otherwise, eventually, we will get two adjacent nodes in a cycle of length 5, such that there is an edge between them (in opposite directions) and v_6. (This is similar to the argument that we used in Lemma 1.) Thus, the first player wins.

The same strategy will work for any $n \geq 6$. Even though it is hard to test whether or not there exists a Hamiltonian cycle in a directed graph [**?**], it is easier to test whether a tournament has a Hamiltonian cycle; hence deciding who has won the game is easier. It is ironical that conjecture about indegree and outdegree is false for $n > 5$ and yet the first player wins, whereas the conjecture is true for $n < 5$ and the first player loses.

4 Why the Strategy works

In this section, we will try to prove why the first player wins, if he adopts a strategy of section 3.

Claim 1: In a game with five nodes, the first player whose aim is to create a Hamiltonian cycle always wins.

Proof: The first player's strategy is to make at least one node have indegree and outdegree >0 in each of his turns after the first turn. He has five turns to play. In four turns, he has to make five nodes with indegree and outdegree >0. In one of his turns, he will be able to make two nodes with indegree and outdegree > 0. This is as per rule two in his strategy. This rule will (clearly) be satisfied during one of his turns. (Let the first player draw an edge from node 1 to node 2. If the second player draws an edge from node 3 to node 4 then the first player adds the edge from node 2 to node 3, and two nodes, namely 2 and 3 have indegree and outdegree > 0. If the second player draws an edge from node 3 to node 2, then the first player draws an edge from node 2 to node 4. This makes node 2 have indegree and outdegree > 0. Now no matter what the second player does, the first player can make two nodes have indegree and outdegree > 0 in his next turn.)

Claim 2: In a game with five nodes, in which the roles of the players are reversed (i.e., the first player's aim is to prevent a Hamiltonian cycle and the second player's aim is to create a Hamiltonian cycle), then the second player wins.

Proof: The second player's strategy will be the same as the first player's strategy of the regular game (i.e., the one given earlier). The second player's strategy is to make at least one node (with indegree and outdegree > 0) in each of his five turns to play. This will be achieved as per rules 1, 2 and 3 in his strategy.

Claim 3: In a game with six nodes, the first player (the one who creates a Hamiltonian cycle) wins.

Proof: The first player picks a node, say 6, as the distinguished node. His first move is to draw an edge from node 1 to node 6. Afterwards, whenever, the second player draws an edge to (from) node 6, the first player draws an edge from node (to) 6. It is clear (as the degree of node 6 can be at most 5), the second player eventually has to draw an edge among the nodes $\{1, 2, 3, 4, 5\}$. Then the first player adopts the five-node game strategy with these five nodes, in which he plays second. This will guarantee that he creates a cycle of length 5, among nodes $\{1, 2, 3, 4, 5\}$. Since there is at least one edge coming from node 6 and an edge going to node 6, and we have a tournament of six nodes, there is a Hamiltonian cycle. Hence the first player wins.

Claim 4: In a game with six nodes, in which the roles of the players are reversed, the second player wins.

Proof: When the first player draws an edge, e, then the second player designates the node that was not touched by the edge e as a distinguished node. He simply adopts the strategy of playing on the five nodes as the second player. Clearly the second player wins.

Claim 5: In a game with n $>$ 4 nodes, the player whose aim is to create a Hamiltonian cycle wins.

Proof: By induction on the number of nodes. The base case of five nodes is clear from Claims 1 and 2. Assume that the hypothesis is true for $k = n - 1$; Now for $k = n$, consider two cases: The first player has to create a Hamiltonian cycle. The first player draws an edge to the distinguished node, say n. Now if the second player draws an edge in the remaining nodes, the first player creates a cycle of length $n - 1$ by the hypothesis by adopting the strategy of the second player (who creates a Hamiltonian cycle) for n-1 node game. If the second player draws an edge to (from) node n, the first player draws an edge from (to) node n. Eventually, the first player wins by adapting the strategy for n-1 node game, and by the hypothesis, he can create a cycle of length $n-1$. If the second player has to create a Hamiltonian cycle, then when the first player draws an edge e, the second player chooses a node not touched by e as the distinguished node. The second player then adopts a strategy on the $n - 1$ nodes and by hypothesis he can create a cycle of length $n-1$. By his strategy on the distinguished node, he can make sure that there is an edge to and from the distinguished node and hence, in the resulting tournament there will be a Hamiltonian cycle.

5 Conclusion

This game could be extended to play in a complete bipartite graph (i.e., a team tournament). The first player has again a winning strategy, but the analysis becomes complicated.

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