Visual Techniques For Computing Polyhedral Volumes

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Abstract

The volume of regular polyhedra have been a source of interest to geometers since the time of Plato and Aristotle, and formulae for computing the volumes of the dodecahedron and icosahedron can be traced back to ancient Greece. In this paper, we revisit these volumes from a slightly different perspective —we illustrate various constructions that permit the final formulae to be derived by simple visual inspection.

In presenting these techniques, we gain a fresh perspective on the relationship between the dodecahedron, icosahedron, cube, and the golden ratio ϕ. The visual nature of these computational techniques in combination with zome models make these proofs easily accessible.

This paper was authored and typeset using the rich suite of Open Source tools available on the Linux desktop. At a time when most of the attention around Open Source Software is focused on the operating system, we would like to draw readers' attention to the wonderful array of high-quality Open Source authoring and document preparation tools created over the last 25 years by the (LA)TEX community. Figures in this paper were drawn using declarative authoring packages that enabled the first author to draw reliably without having to look at the final output. The high-level markup also makes this content long-lived.

1 Introduction

The volume of regular polyhedra have been a source of interest to geometers since the time of Plato and Aristotle, and formulae for computing the volumes of the dodecahedron and icosahedron can be traced back to ancient Greece —[Coe89, Coe73, Wei98]. In this paper, we revisit these volumes from a slightly different perspective. We illustrate various constructions that permit the final formulae to be derived by simple visual inspection.

In presenting these techniques, we gain a fresh perspective on the relationship between the dodecahedron, icosahedron, cube, and the golden ratio ϕ. All of the techniques described in this paper are made tangible using Zome systems' polyhedra building kit. The visual nature of our computational technique along with zome models make these proofs accessible to students of all levels.

The techniques described in this paper demonstrate the value of picking an appropriate frame of reference when solving problems in mathematics. Using Zome Systems building kit, our basic units of measure are 1, $\sin 60$, $\sin 72$, $\sqrt{2}$, $\frac{1}{\sqrt{2}}$ and these same lengths scaled by powers of the Golden Ratio ϕ. These units make for easy computation of expressions that would otherwise be computationally awkward. In this context, selecting the right basic units of measure is equivalent to picking an appropriate coordinate system or equivalently, the right set of basis vectors.

1.1 Basic Lengths

Zome Systems —http://www.zometool.com—([HP01]) leverages the following mathematical facts:

- The symmetry of the dodecahedron and icosahedron.
- The golden ratio and its *scaling* property.

Nodes in the zome kit have all the *directions* needed to build the various Platonic solids. Zome struts come in four different colors (blue, red, yellow, green) with struts of a given color corresponding to a given symmetry axis. Struts of each color come in three different sizes, and successive struts of the same color are in the *golden ratio*. In the rest of this paper, we refer to zome struts by symbols made up of the first letter of the color, suffixed by $(1, 2, \ldots)$; see table 1 on the following page for a table of all the zome lengths. The table also shows the mathematical significance of these lengths in brief; the rest of the paper builds on these properties.

1.2 Identities Of The Golden Ratio

This section derives some useful identities involving the golden ratio ϕ and the basic lengths in the zome system.

Color	Significance	1	2	3
Blue	Unity	1	ϕ	ϕ^2
Red	Radius of I_1	$\sin 72$	$\phi \sin 72$	$\phi^2 \sin 72$
Yellow	Radius of C_1	$\sin 60$	$\phi \sin 60$	$\phi^2 \sin 60$
Green	Face diagonal of C_1	$\sqrt{2}$	$\phi\sqrt{2}$	$\phi^2\sqrt{2}$
green	Radius of a C_1 face	$\frac{1}{\sqrt{2}}$	$\phi\frac{1}{\sqrt{2}}$	$\phi^2\frac{1}{\sqrt{2}}$

Table 1: Basic zome lengths. Zome struts come in different colors and sizes. The sizes are mathematically significant. Here, I_1 denotes the unit icosahedron and C_1 denotes the unit cube.

Successive Powers

Successive powers of the golden ratio form a Fibonacci sequence.

$$1 + \phi = \phi^2 \tag{1.1}$$
$$\phi + \phi^2 = \phi^3 \tag{1.2}$$
$$\vdots = \vdots$$
$$\phi^{n-2} + \phi^{n-1} = \phi^n \tag{1.3}$$

Golden ratio and the diagonal of the pentagon.

Observe one of the small isosceles triangles in figure 1 on the next page —it has base $B_2 = \phi$ and sides $B_1 = 1$. Dropping a perpendicular from the apex of this triangle to its base bisects its base $B_2 = \phi$. From the right-triangles that result, we get

$$2 \cos 36 = \phi \tag{1.4}$$

Identity of the golden rhombus.

Observing that R_1 lines constitute the radii of a golden rectangle gives R_1 in terms of the golden ratio. R_1 can be computed by observing the right-triangle marked ABC in figure 2 on the following page.

$$1 + \phi^2 = 4{R_1}^2 \tag{1.5}$$

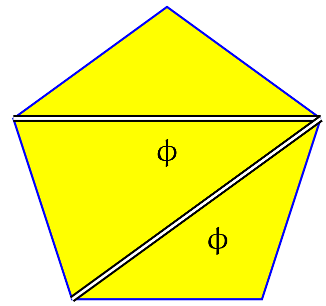

Figure 1: Diagonal of a unit pentagon has length ϕ. Drawing 2 diagonals incident on a vertex of the pentagon divides it into 3 triangles, and the pentagonal area can be computed by summing these triangles to get $\sin 72(1 + \frac{\phi}{2})$.

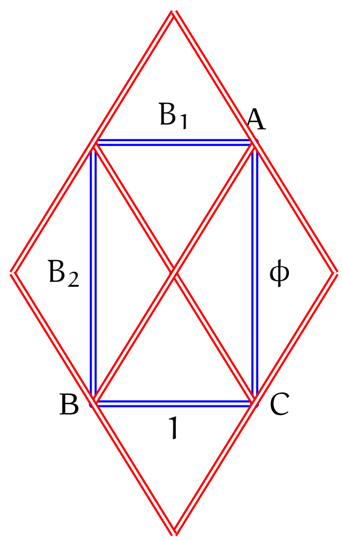

Figure 2: Relation between the golden rectangle and golden rhombus. Observe right-triangle marked ABC to derive the identity given by equation (1.6).

R_1 in terms of trigonometric ratios

The length of R_1 can be expressed in terms of trigonometric ratios by putting together the identities derived so far.

$$\cos 36 = \frac{\phi}{2} \tag{1.6}$$
$$\cos 2 * 18 = 2\cos^2 18 - 1 \tag{1.7}$$
$$= 2\sin^2 72 - 1 \tag{1.8}$$

Combining these gives

$$\sin^2 72 = \frac{1}{2} + \frac{\phi}{4} \tag{1.9}$$
$$= \frac{2+\phi}{4} \tag{1.10}$$
$$= \frac{1+\phi^2}{4} \tag{1.11}$$
$$= R_1^2 \tag{1.12}$$

Powers of the golden ratio and trigonometry.

Construct a right-triangle of sides B_1 and B_3 —its hypotenuse is the result of joining the B_1 and B_3 lines using $2Y_2 = \phi\sqrt{3}$ struts —see figure 3 on the next page. This gives the identities

$$1 + \phi^4 = 3\phi^2 \tag{1.13}$$
$$\frac{1}{\phi^2} = 3 - \phi^2 \tag{1.14}$$

Computing other trigonometric ratios

From the identities for $\sin 72$ and $\cos 36$, we can derive $\sin 36$ as follows:

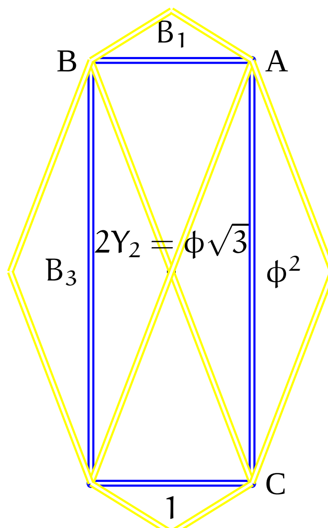

Figure 3: This figure shows a Y_2 yellow rhombus and its relation to a rectangle of sides $B_1 \times B_3$ rectangle. It is used in deriving the identity shown in equation (1.13) —observe that this figure is the same as figure 2 on page 4 scaled by a factor of ϕ in the Y direction.

$$2 \sin 36 \cos 36 = \sin 72 \tag{1.15}$$

$$2 \cos 36 = \phi \tag{1.16}$$

$$\sin 36 = \frac{\sqrt{1+\phi^2}}{2\phi} \tag{1.17}$$

2 Locating Vertices Of Various Polyhedra

This section locates the vertex coordinates of regular polyhedra using zome models. Constructions used in this exercise prove useful in computing distances needed for the volume computations.

2.1 Locating Vertices Of The Cube

Construct a unit cube of side $B_1 = 1$. Let $(0,0,0)$ be the center of the cube. The coordinates of the cube vertices are

$$\{(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\}.$$

Notice that by the choice of the basic zome lengths, the center of the cube can be connected to the 8 cube vertices by Y_1 struts. Thus, the radius of the unit cube is $\frac{\sqrt{3}}{2} = \sin 60 = Y_1$.

2.2 Locating Vertices Of The Tetrahedron

Construct a unit B_1 cube. Pick the cube vertex that lies in the first octant having coordinates $T = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Draw the 3 face diagonals of the cube incident on vertex T. Each diagonal has length $G_1 = \sqrt{2}$. Finally, draw the face diagonals of the cube that connect the end-points of the 3 diagonals just drawn. This constructs a tetrahedron of side $G_1 = \sqrt{2}$ inside the B_1 cube.

Let $(0,0,0)$ be the center of the cube —notice that it is also the center of the G_1 tetrahedron. The vertex coordinates of this G_1 tetrahedron can be read from this model as:

$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$
$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$

As shown in section 2.1, the center of the above model can be joined to the cube vertices using $Y_1 = \sin 60$ yellow lines. From this it follows that the center of the G_1 tetrahedron can be joined to the tetrahedral vertices using $Y_1 = \sin 60$ lines. Observe that drawing these radii divides the interior of the tetrahedron into 4 congruent pyramids with a G_1 equilateral triangle as base and vertical sides Y_1.

2.3 Locating Vertices Of The Octahedron

Construct an octahedron of side $G_1 = \sqrt{2}$. View this model placed on one of its vertices so that the opposite vertex is *directly above* the chosen vertex. Let $(0,0,0)$ be the center of the octahedron. Let the symmetry axis formed by joining the *top* and *bottom* vertices denote the Z axis. Connect these vertices to the center using $B_1 = 1$ struts. Similarly, locate opposite pairs of vertices and draw the X and Y axes. From this model, the vertex coordinates of the octahedron are given by:

$$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.$$

2.4 Locating Vertices Of The Rhombic Dodecahedron

Construct a B_1 unit cube. Construct pyramids of side $Y_1 = \sin 60$ on each face of this cube. Removing the cube edges leaves a rhombic dodecahedron of side Y_1. The rhombic dodecahedron has 12 faces and 14 vertices.

Vertices of the rhombic dodecahedron can be categorized as:

- Vertices of the unit cube.
- The apex of each of the 6 pyramids described above.

Let $(0,0,0)$ be the center of this rhombic dodecahedron —hence the center of the cube. This gives the coordinates of the 8 of the 14 vertices to be

$$(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}).$$

Next, observe that by construction, the Y_1 pyramids built on each face are congruent to the pyramids constructed by connecting the center of the unit cube to its vertices —see section 2.1 on the page before. Thus, the height of these pyramids is half the side of the cube and therefore $\frac{1}{2}$.

The coordinates of these 6 vertices of the rhombic dodecahedron are then given by

$$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.$$

Notice that from the above, the rhombic dodecahedron has all the vertices of a B_1 cube and the dual G_1 octahedron —a fact that will be used later in the techniques for computing the volume of the rhombic dodecahedron.

2.5 Locating Vertices Of The Cube-octahedron

The cube-octahedron is the dual to the rhombic dodecahedron described in section 2.4 on the preceding page. The rhombic dodecahedron was shown to have the vertices of the cube and the octahedron; by duality, the cube-octahedron has the faces of the cube and the octahedron.

Construct a $2B_1$ cube. Join the mid-points of adjacent edges using $G_1 = \sqrt{2}$ green lines. Removing the cube edges leaves a G_1 cube-octahedron.

Let $(0, 0, 0)$ be the center of the $2B_1$ cube, hence the center of the cube-octahedron. From this model, the vertex coordinates can be read as:

$$\{(0, \pm 1, \pm 1), (\pm 1, 0, \pm 1), (\pm 1, \pm 1, 0)\}.$$

2.6 Locating Vertices Of The Dodecahedron

Consider the unit dodecahedron with sides B_1. Each face is a unit pentagon. A diagonal of a pentagon is in the golden ratio to the length of its side; consequently, any face diagonal can be drawn by using a B_2 strut —see figure 1 on page 4. Place this dodecahedron on one of its edges; this *base edge* can be connected to its opposite (*top*) edge using a pair of B_3 struts to form a $B_3 \times B_1$ rectangle. View this model with the $B_3 \times B_1$ rectangle just constructed lying in the ZX plane. Locate opposing pairs of edges of the dodecahedron to similarly construct $B_3 \times B_1$ rectangles in the XY and YZ planes. This consumes 12 of the 20 vertices.

View this model with the XY, YZ and ZX planes in place; the remaining 8 vertices form a cube whose sides are face diagonals of the dodecahedron and therefore of length $B_2 = \phi$.

Let $(0, 0, 0)$ be the center of the dodecahedron. Then the 8 vertices making up the cube have coordinates $(\pm\frac{\phi}{2}, \pm\frac{\phi}{2}, \pm\frac{\phi}{2})$.

The 4 vertices of the $B_3 \times B_1$ rectangle in the XY plane have coordinates $(\pm\frac{\phi^2}{2}, \pm\frac{1}{2}, 0)$. The remaining coordinates can be located in an analogous manner by examining the rectangles in the ZX and YZ planes —notice that these are just rotated copies of the $B_3 \times B_1$ rectangle in the XY plane —see figure 4 on the following page for a view of one face of the dodecahedron from this model.

2.7 Locating Vertices Of The Icosahedron

Construct a unit icosahedron of side B_1. Place it on one of its edges, and notice that the *bottom edge* can be connected to its opposite (*top*) edge using a pair of $B_2 = \phi$ struts. View this model so the $B_2 \times B_1$ golden rectangle just constructed lies in the ZX plane. Locate opposing pairs of edges to construct $B_2 \times B_1$ rectangles in the XY and YZ planes. This accounts for the 12 vertices of the icosahedron.

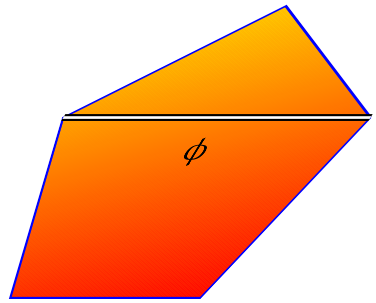

Figure 4: This view shows the three-dimensional perspective of a face of the dodecahedron with one of its face diagonals. The face is tilted by 58.282525589 deg $= \arctan \phi$.

Let $(0,0,0)$ be the center of the icosahedron. The 4 vertices of the $B_2 \times B_1$ rectangle in the XY plane have coordinates $(\pm\frac{\phi}{2}, \pm\frac{1}{2}, 0)$. The coordinates of the remaining 8 vertices of the icosahedron can be similarly read off the model —notice that they are the appropriate rotation of the 4 vertices shown above. Thus, we get the coordinates for the 12 vertices of the icosahedron to be:

$$\boxed{\begin{array}{l} (\pm\frac{\phi}{2}, \pm\frac{1}{2}, 0) \\ (0, \pm\frac{\phi}{2}, \pm\frac{1}{2}) \\ (\frac{1}{2}, 0, \pm\frac{\phi}{2}) \end{array}} \tag{2.1}$$

Notice that this construction also reveals that a unit icosahedron can be *packed* inside a cube of side ϕ —a fact that will be used later in one of the techniques for computing the volume of the icosahedron.

As shown in the table of basic lengths (see table 1 on page 3), the R_1 zome strut is the radius of the B_1 icosahedron. Observe that these 12 radii draw the diagonals of the $B_2 \times B_1$ golden rectangles in the XY, YZ and ZX planes in the above model.

2.8 Locating Vertices Of The Rhombic Triacontahedron

The rhombic triacontahedron is an Archimedian polyhedron with 30 faces and 32 vertices. In locating its vertices, we will see that it has the vertices of the dodecahedron and its dual icosahedron.

Construct a B_1 dodecahedron. Construct red pyramids of vertical side $R_1 = \sin 72$ on each of the faces of the dodecahedron. The edges of the dodecahedron form the short diagonal of each rhombic face —removing these edges of the dodecahedron leaves a red rhombic triacontahedron of side $R_1 = \sin 72$.

Next, observe that the long diagonal of each rhombic face can be drawn using $B_2 = \phi$. Thus, the diagonals of the rhombic faces are in the golden ratio and each face of the rhombic triacontahedron is a golden rhombus. Finally, observe that drawing the B_2 diagonals and removing the red edges would leave a B_2 icosahedron. See figure 2 on page 4 for an illustration showing the relation between the golden rhombus and the fact that the radii of a B_1 icosahedron draw the diagonals of a golden rectangle as described in 2.7 on page 9.

2.9 Locating Vertices Of The Icosidodecahedron

The icosidodecahedron has 32 faces and 30 vertices, and is dual to the rhombic triacontahedron described in 2.8 on the preceding page. Since the rhombic triacontahedron has all the vertices of the dodecahedron and its dual icosahedron, by duality it follows that the icosidodecahedron has all the faces of the dodecahedron and icosahedron.

Observe that a zome node has 30 holes that can take blue struts. Inserting B_2 struts into each of these 30 holes and connecting their end-points with B_1 struts constructs an B_1 icosidodecahedron. By construction, the radius of the icosidodecahedron of side $B_1 = 1$ is $B_2 = \phi$.

Let $(0, 0, 0)$ be the center of the icosidodecahedron. Place this model on one of its vertices, and observe that the B_2 struts connecting the center to the *bottom* and *top* vertices can be viewed as the Z axis. Locate the X and Y axes in a similar manner to see that 6 of the 30 vertices of the icosidodecahedron are also vertices of an octahedron of side $G_2 = \phi\sqrt{2}$. This gives 6 of the 30 vertices to be

$$\{(\pm\phi, 0, 0), (0, \pm\phi, 0), (0, 0, \pm\phi)\}.$$

We obtained the above fact by placing the icosidodecahedron on *any* one of its vertices — it therefore follows that the remaining 24 vertices can in turn be divided into disjoint sets of 6 vertices each, with each set corresponding to the vertices of a rotated copy of the $G_2 = \phi\sqrt{2}$ octahedron.

This leads to an important result —the unit icosidodecahedron can be *wrapped around* the compound of 5 concentric octahedra of side $\phi\sqrt{2}$.

To compute the coordinates of the remaining vertices of the icosidodecahedron, consider the model built in section 2.6 on page 9 where we constructed a B_1 icosahedron. Scale this model by 2 to obtain a $2B_1$ icosahedron.

Let $(0, 0, 0)$ be the center of this model. By scaling all values computed in equation (2.1), we first locate the 12 vertices of the $2B_1$ icosahedron to be:

$$\begin{array}{|c|}\hline (\pm\phi, \pm 1, 0) \\ (0, \pm\phi, \pm 1) \\ (\pm 1, 0, \pm\phi) \\ \hline \end{array} \quad (2.2)$$

Next, observe the $2B_1 \times 2B_2$ golden rectangles in the XY, YZ and ZX planes, and consider the mid-points of the $2B_1$ sides. These have coordinates $\{(\pm\phi, 0, 0), (0, \pm\phi, 0), (0, 0, \pm\phi)\}$. Thus, the mid-points of 6 of the 30 edges of the $2B_1$ icosahedron give the vertices of the G_2 octahedron. By symmetry, it follows that the 30 edges of the $2B_1$ icosahedron can be partitioned into 5 disjoint sets of 6 edges each, where the mid-points of edges in any given partition form a rotated copy of a G_2 octahedron.

By combining the above with the earlier result that the vertices of the icosidodecahedron are the same as the vertices of the compound of 5 concentric G_2 octahedra, we can compute the coordinates of all 30 vertices by reading off the mid-points of the 30 edges of the $2B_1$ icosahedron.

Observe that by construction the $2B_1$ icosahedron as oriented is symmetric about the coordinate axis. Therefore, we need only compute the coordinates of the remaining 24 vertices in one of the octants.

Consider the 3 vertices of the icosidodecahedron in the first octant. By construction, these are the mid-points of the sides of the $2B_1$ triangle shown in figure 5 on the following page.

The coordinates of of the 30 vertices of the icosidodecahedron are therefore:

$$\begin{array}{|l|}\hline (\pm\phi, 0, 0) \quad (0, \pm\phi, 0) \quad (0, 0, \pm\phi) \\ \hline (\pm\frac{\phi}{2}, \pm\frac{1+\phi}{2}, \pm\frac{1}{2}) \\ (\pm\frac{1+\phi}{2}, \pm\frac{1}{2}, \pm\frac{\phi}{2}) \\ (\pm\frac{1}{2}, \pm\frac{\phi}{2}, \pm\frac{1+\phi}{2}) \\ \hline \end{array} \quad (2.3)$$

Finally, observe that this construction has shown how the icosidodecahedron can be *wrapped around* the compound of 5 concentric octahedra. Applying duality to this result, and using the fact that:

- Vertices map to faces in the dual.
- The *inside* and *outside* reverse roles in the dual.

the rhombic triacontahedron which is dual to the icosahedron can be seen to have each of its 30 rhombic faces on each of the 30 cube faces of the compound of 5 concentric cubes.

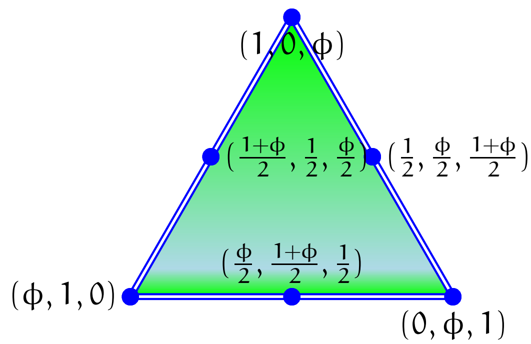

Figure 5: Face of the $2B_1$ icosahedron in the first octant. Its mid-points give 3 vertices of the icosidodecahedron of side B_1, and by symmetry,, these help locating the vertices of the icosidodecahedron that do not lie on the coordinate axes.

3 Using The Cube To Compute Volumes

3.1 Volume Of The Tetrahedron

Consider the G_1 tetrahedron constructed in section 2.2 on page 7. From this model, the interior of the B_1 cube can be decomposed into a G_1 tetrahedron and 4 pyramids having a green base and blue vertical sides. Place the model on one of the cube faces, and observe one of these pyramids. It has a right triangle of sides B_1, B_1, G_1 as base, and a side of the B_1 cube as its height. This gives the volume of this pyramid to be:

$$\frac{1}{2}\frac{1}{3} = \frac{1}{6}.$$

Subtracting 4 copies of this pyramid from the cube gives the volume of the G_1 tetrahedron V_T to be:

$$\begin{aligned} V_T &= 1^3 - \frac{4}{6} \\ &= \frac{1}{3}. \end{aligned} \tag{3.1}$$

3.2 Volume Of The Octahedron

Consider the G_1 octahedron constructed in section 2.3 on page 8. Place it on one of its triangular faces. Construct a G_1 tetrahedron and observe that the tetrahedron has the *same* face as the octahedron. Take 4 copies of this G_1 tetrahedron, and place them on 4 faces of the octahedron to form a $2G_1$ tetrahedron. This shows that the $2G_1$ tetrahedron can be decomposed into an octahedron and 4 tetrahedra.

We computed the volume of the G_1 tetrahedron to be $\frac{1}{3}$ in section 3.1. By applying the scaling rule, the volume of the $2G_1$ tetrahedron is $\frac{8}{3}$. Subtracting 4 copies of the G_1 tetrahedron from the $2G_1$ tetrahedron gives the volume of the G_1 octahedron V_O:

$$\begin{aligned} V_O &= \frac{8}{3} - 4\frac{1}{3} \\ &= \frac{4}{3}. \end{aligned} \tag{3.2}$$

3.3 Volume Of The Rhombic Dodecahedron

Consider the rhombic dodecahedron constructed in section 2.4 on page 8. Its volume can be decomposed into the unit cube and 6 pyramids having a B_1 square base and Y_1 vertical sides.

Section 2.4 on page 8 also showed the height of this pyramid to be $\frac{1}{2}$. This gives the volume of this pyramid V_P to be:

$$\begin{aligned} V_P &= \frac{1}{2}\frac{1}{3} \\ &= \frac{1}{6}. \end{aligned} \tag{3.3}$$

The volume of the rhombic dodecahedron V_{RD} is therefore:

$$\begin{aligned} V_{RD} &= 1 + 6\frac{1}{6} \\ &= 2 \\ &= \text{Twice the volume of the unit cube.} \end{aligned} \tag{3.4}$$

This can also be seen by realizing that the yellow pyramid constructed on each face of the B_1 cube is congruent to the pyramid constructed by joining the center of the cube to the vertices of a given face. Thus, the 6 pyramids constructed *outside* the cube can be packed into the interior of the cube, giving the volume of the Y_1 rhombic dodecahedron to be twice the volume of the unit cube.

Finally, consider once again the model of the rhombic dodecahedron of side $\sin 60$ and draw the longer diagonal of each face. By the choice of zome lengths, this is a $G_1 = \sqrt{2}$ green line. Drawing the long diagonal of all 12 faces gives a G_1 octahedron. This decomposes the rhombic dodecahedron into an octahedron and 8 pyramids having a G_1 equilateral triangle as base and Y_1 vertical sides. We showed in section 2.2 on page 7 that this pyramid is $1/4$ the volume of the G_1 tetrahedron. From this we can compute the volume of the rhombic dodecahedron V_{RD} to be

$$\begin{aligned} V_{RD} &= \frac{4}{3} + 8\frac{1}{3}\frac{1}{4} \\ &= 2. \end{aligned} \tag{3.5}$$

See figure 6 on the next page for a visual representation of this relationship.

3.4 Volume Of The Cube-octahedron

Consider the cube-octahedron constructed in section 2.5 on page 9. This shows that the $2B_1$ cube can be decomposed into a cube-octahedron and 8 pyramids. Notice that these pyramids are the

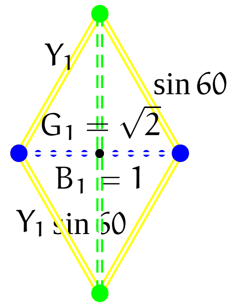

Figure 6: The yellow rhombus of side $Y_1 = \sin 60$ has a short $B_1 = 1$ diagonal and a long $G_1 = \sqrt{2}$ diagonal. This relation leads to two equivalent ways of constructing a rhombic dodecahedron of side Y_1.

same that occurred in section 3.1 on page 14 while computing the volume of the tetrahedron —we computed this to be $\frac{1}{6}$. Thus, the volume of the cube-octahedron V_{CO} is:

$$\begin{aligned} V_{CO} &= 8 - 8\frac{1}{6} \\ &= \frac{20}{3}. \end{aligned} \tag{3.6}$$

4 Volume Of A Dodecahedron

Consider the model built in section 2.6 on page 9 in locating the vertices of the dodecahedron. A dodecahedron can be viewed as the result of adding 6 *roof* structures to a cube —this construction was known to Euclid. This decomposition of the dodecahedron can be used in computing its volume.

From the model built in 2.6 on page 9, the cube has sides $B_2 = \phi$ and therefore has volume ϕ^3. It only remains to compute the volume of the *roof* structures.

Each *roof* has a square base of side B_2. The vertical faces are a pair of triangles and trapezium. Consider one of these trapezoidal faces; the parallel sides have length $B_2 = \phi$ and $B_1 = 1$.

A trapezium can be viewed as the sum of a triangle and a parallelogram. Applying this decomposition to the *roof* structure, it can be decomposed into a pyramid and a triangular cross-section.

Pyramid volume.

This pyramid has a rectangular base with sides B_2 and $B_2 - B_1$. Applying identities of the golden ratio, $B_2 - B_1 = \phi - 1 = \frac{1}{\phi}$, giving the area of the base A_{base} to be

$$\begin{aligned} A_{base} &= \phi \frac{1}{\phi} \\ &= 1. \end{aligned} \tag{4.1}$$

From the model constructed in 2.6 on page 9, the height of this pyramid is half of $B_3 - B_2$. Since successive zome lengths are in the golden ratio,

$$\frac{B_3 - B_2}{2} = \frac{1}{2}. \tag{4.2}$$

The volume of this pyramid using (4.1) and (4.2) is therefore:

$$\begin{aligned} V_P &= 1 \times \frac{1}{2}\frac{1}{3} \\ &= \frac{1}{6}. \end{aligned} \tag{4.3}$$

Next, consider the triangular cross-section. Its face is a triangle of side B_2 whose height is the same as the height of the pyramid computed above in equation (4.2). The area of the triangular face is therefore $\frac{\phi}{4}$; the length of the cross-section is $B_1 = 1$, giving its volume to be

$$\frac{\phi}{4}.$$

The dodecahedron as constructed is equal to the cube plus 6 *roof* structures —one on each face of the cube. Thus, the volume of the dodecahedron is

$$V_D = \phi^3 + 6(\frac{\phi}{4} + \frac{1}{6}). \tag{4.4}$$

Radius Of The Dodecahedron

Consider once again the cube of side B_2 identified in the model built in 2.6 on page 9. As shown in 1 on page 3, the radius of this cube is $Y_2 = \phi \sin 60$. We identified the vertices of this cube of side $B_2 = \phi$ by first placing the dodecahedron on *any* one of its edges. Therefore there is nothing *special* about these 8 of the 20 vertices of the dodecahedron, and it follows by symmetry that all vertices of the dodecahedron are a distance $Y_2 = \phi \sin 60$ from its center. Notice that drawing these 20 radii decomposes the interior of the dodecahedron into 12 congruent pyramids that have a unit pentagon as the base and Y_2 as the vertical sides. This construction in turn leads to an alternative technique for computing the volume of the dodecahedron.

5 Volume Of The Icosahedron

The icosahedron is the *dual* to the dodecahedron. This duality when applied to the technique described in the previous section leads to a solution for computing the volume of the icosahedron.

5.1 Volume Of The Icosahedron Part I

We computed the volume of the dodecahedron by building a cube *inside* the dodecahedron. The dual to this solution is to build an octahedron (dual to the cube) *around* the icosahedron. Observe that when we take the dual the *inside* comes *out*.

View the model built in 2.7 on page 9 placed on one of the icosahedral edges and oriented so the $B_2 \times B_1$ rectangles lie in the XY, YZ and ZX planes. Construct a right-triangle with the *top* B_1 edge of the icosahedron as its hypotenuse in the ZX plane —in the zome model, this triangle has $g_1 = \frac{1}{\sqrt{2}}$ green *legs*. Thus, this right-triangle has $B_1 = 1$ as the hypotenuse and sides $g_1 = \frac{1}{\sqrt{2}}$. Repeat this construction on the *bottom* B_1 edge. Finally, construct two more right-triangles each with one of the $B_2 = \phi$ sides as the hypotenuse. These right-triangles have sides $g_2 = \frac{\phi}{\sqrt{2}}$. The above constructs a square of side $\frac{1+\phi}{\sqrt{2}}$ around the golden rectangle $B_2 \times B_1$ in the ZX plane —see figure 5.1 on the following page.

Repeat this construction for the $B_2 \times B_1$ rectangles in the XY and YZ planes. The result is to construct an octahedron of side

$$\begin{aligned} g_1 + g_2 &= \frac{1+\phi}{\sqrt{2}} \\ &= \frac{\phi^2}{\sqrt{2}}. \end{aligned}$$

The volume of an octahedron of side $\sqrt{2}$ is $\frac{4}{3}$ as shown in 3.2 on page 14. The volume of the octahedron of side $\frac{\phi^2}{\sqrt{2}}$ constructed above has its side scaled by $\frac{\phi^2}{2}$ and its volume by the scaling rule is:

$$\begin{aligned} V_O &= \frac{\phi^6}{8}\frac{4}{3} \\ &= \frac{\phi^6}{6}. \end{aligned} \tag{5.1}$$

Next, we compute the volume of the *pyramids* we added to the icosahedron in constructing the octahedron. View the model of the octahedron around the icosahedron with the $B_2 \times B_1$ rectangles lying in the XY, YZ and ZX planes. Observe one of the g_1, g_1, B_1 right-triangle in the XY plane, and consider the *obtuse pyramid* that has this triangle as its base. The apex of this

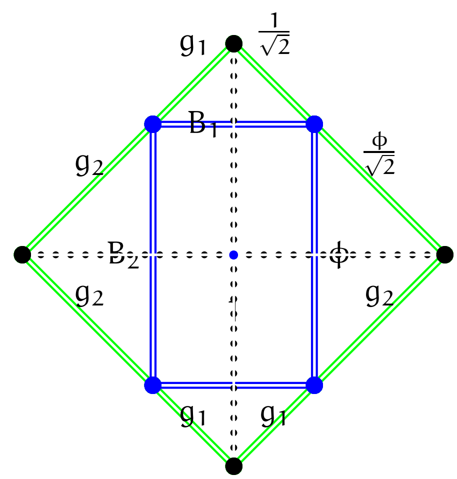

Figure 7: This figure shows a green square constructed around a blue golden rectangle.

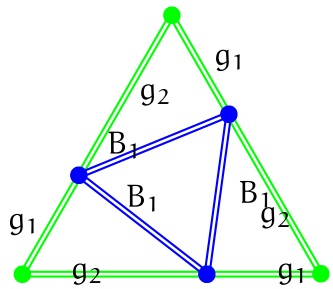

Figure 8: This figure shows a face of the compound of the octahedron and icosahedron constructed in section 5.1 on page 18. The green triangle is the octahedral face, and the embedded blue triangle is a face of the icosahedron. The blue triangle divides the sides of the green triangle in the golden ratio.

pyramid is a vertex of the *top* edge of the icosahedron with Z coordinate $\frac{\phi}{2}$ which is also the *height* of this pyramid. This gives the volume of the obtuse pyramid to be:

$$\begin{aligned} V_P &= \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{3}\frac{\phi}{2} \\ &= \frac{\phi}{24}. \end{aligned} \tag{5.2}$$

There are two copies of this pyramid at each of the 6 vertices of the octahedron. Thus, the volume of the icosahedron is:

$$\begin{aligned} V_I &= \frac{\phi^6}{6} - 12\frac{\phi}{24} \\ &= \frac{5\phi^2}{6} \qquad \text{(using (1.1))} \end{aligned} \tag{5.3}$$

5.2 Volume Of The Icosahedron Part II

The volume of the unit icosahedron can also be computed by *packing* it in a B_2cube, and subtracting the volume of the space between the cube and the icosahedron from the volume of the cube.

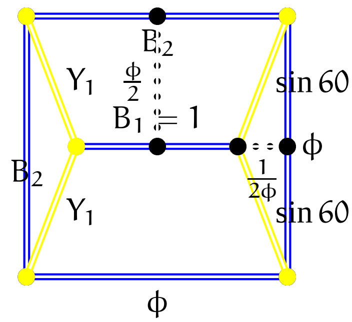

Figure 9: One face of the B_2 cube with the contained B_1 icosahedral edge. The icosahedral edge is connected to the vertices of the surrounding B_2 square using $Y_1 = \sin 60$ lines. The vertical dotted lines show the height of the trapezoidal base; the horizontal dotted lines show the height of the trapezoidal pyramid.

Place the B_1 icosahedron on one of its edges as before. We will construct a B_2 cube around this icosahedron so that each face of the cube contains a corresponding edge of the icosahedron —so for instance, the *top* edge of the icosahedron lies within the *top* face of this cube. Figure 9 shows one such face of the cube along with the contained icosahedral edge.

For each face of the B_2 cube, join the vertices of the contained icosahedral edge to the vertices of that face using $Y_1 = \sin 60$ lines as shown in figure 9. This constructs yellow *triangular pyramids* with vertical side Y_1 on 8 faces of the icosahedron, one per vertex of the cube. Consider one such pyramid; its base is a face of the icosahedron and therefore a B_1 equilateral triangle, and its apex the corresponding vertex of the cube. The vertical sides of this pyramid are the yellow lines shown in figure 9 and have length $Y_1 = \sin 60$.

Observe further that for each face of the cube, there are two trapezoidal pyramids inside the cube. These have the B_1, Y_1, B_2, Y_1 trapezium appearing as part of figure 9 as base. Consider one of the trapezoidal pyramid with its base contained in the *front* face of the B_2 cube —for now, consider the trapezoidal pyramid whose base appears in the bottom half of figure 9. The apex of this pyramid then lies on the bottom face of the cube. The height of this pyramid is the height of the Y_1, Y_1, B_1 triangle with base B_1 —shown in figure 9 on the preceding page by horizontal dotted lines. The height of this trapezoidal pyramid is therefore

$$\frac{B_2 - B_1}{2} = \frac{1}{2\phi}.$$

Now, observe that the volume of the B_2 cube can be decomposed into the following disjoint pieces:

- The volume of the unit icosahedron.
- the 8 triangular pyramids —one for each of the 8 vertices of the cube.
- 12 trapezoidal pyramids 2 for each of the 6 faces of the cube.

Volume of the triangular pyramid.

To calculate the volume of a triangular pyramid, we first compute its height. We do this by applying the Pythogorian theorem to a right-triangle constructed by dropping a perpendicular from the apex of this pyramid to its base —it passes through the *centroid* of the B_1 equilateral triangle. For a unit equilateral triangle, the centroid is at a distance $\frac{1}{\sqrt{3}}$ from its vertices. The right-triangle therefore has base $\frac{1}{\sqrt{3}}$ and and hypotenuse $Y_1 = \frac{\sqrt{3}}{2}$. The height of the pyramid is therefore

$$\begin{aligned} H_P &= \sqrt{\frac{3}{4} - \frac{1}{3}} \\ &= \sqrt{\frac{5}{12}}. \end{aligned} \tag{5.4}$$

The area of the triangular base is

$$\frac{\sqrt{3}}{4}.$$

The volume of the triangular pyramid using (5.4) is:

$$\begin{aligned} V_P &= \frac{1}{3}\frac{\sqrt{3}}{4}\sqrt{\frac{5}{12}} \\ &= \frac{\sqrt{5}}{24}. \end{aligned}$$

Volume of the trapezoidal pyramid.

We first compute the area of the B_1, Y_1, B_2, Y_1 trapezium that is the base of the trapezoidal pyramid —see figure 9 on page 21. The height of this trapezium —shown by the vertical dotted line in figure 9 on page 21—is $\frac{\phi}{2}$. The area of the base trapezium is therefore

$$\begin{aligned} A &= \frac{1}{2}\frac{\phi}{2}(\phi+1) \\ &= \frac{\phi^3}{4}. \end{aligned} \tag{5.5}$$

As shown in figure 9 on page 21, the height of the trapezoidal pyramid is $\frac{1}{2\phi} = \frac{\phi-1}{2}$. Using (5.5) the volume of the trapezoidal pyramid is therefore

$$\begin{aligned} V_P &= \frac{1}{3}\frac{1}{2\phi}\frac{\phi^3}{4} \\ &= \frac{\phi^2}{24}. \end{aligned}$$

Putting the pieces together, we sum the volumes of the 8 triangular pyramids and 12 trapezoidal pyramids to get

$$\begin{aligned} I_R &= \frac{\sqrt{5}}{3} + \frac{\phi^2}{2} \\ &= \frac{2\phi-1}{3} + \frac{\phi^2}{2}. \end{aligned} \tag{5.6}$$

The volume of the icosahedron is given by subtracting the residue shown in equation (5.6) from ϕ^3, the volume of the B_2 cube. Writing $\phi^3 = \phi^2 + \phi$, we get

$$\begin{aligned} V_I &= \phi^2 + \phi - \frac{2\phi-1}{3} - \frac{\phi^2}{2} \\ &= \frac{5\phi^2}{6}. \end{aligned}$$

6 Volume Of The Rhombic Triacontahedron

We showed in section 2.8 on page 10 that the vertices of the rhombic triacontahedron are the vertices of dodecahedron and its dual icosahedron. From the model constructed in section 2.8, the Rhombic triacontahedron consists of a B_2 icosahedron and 20 R_1 pyramids constructed on each of the triangular icosahedral faces. These pyramids have a B_2 equilateral triangle as their base —the vertical sides of the pyramids are the $R_1 = \sin 72$ edges of the rhombic triacontahedron.

The volume of the rhombic triacontahedron can therefore be written as the sum of the volume of the B_2 icosahedron and 20 pyramids.

Volume of a Triangular Pyramid

By following the method used in section 5.2 on page 20, and applying identities of the golden ratio, we compute the height of this pyramid to be

$$\begin{aligned} H_P &= \sqrt{\sin^2 72 - \frac{\phi^2}{3}} \\ &= \sqrt{\frac{1+\phi^2}{4} - \frac{\phi^2}{3}} \qquad \text{(using (1.6))} \\ &= \frac{\sqrt{(3-\phi^2)}}{\sqrt{12}} \\ &= \frac{\sqrt{3}}{6\phi} \qquad \text{(using (1.13))} \\ V_P &= \frac{\phi}{24} \qquad \text{(using the height from the above equation).} \end{aligned}$$

Summing The Parts

The volume of the $B_1 = 1$ icosahedron is $\frac{5\phi^2}{6}$ see section 5.1 on page 18. The volume of the $B_2 = \phi$ icosahedron is obtained by the scaling rule to be:

$$\phi^3 \frac{5\phi^2}{6}.$$

The volume of the 20 triangular pyramids is

$$\frac{5}{6}\phi.$$

Using the identity $\phi^4 + 1 = 3\phi^2$, we get the volume to be:

$$\begin{aligned} V_{RT} &= \frac{5}{6}(\phi^5 + \phi) \\ &= \frac{5\phi}{6}(\phi^4 + 1) \\ &= \frac{5\phi^3}{2} \qquad \text{(Using (1.13))}. \end{aligned}$$

7 Volume Of The Icosidodecahedron

We showed in section 2.9 on page 11 that the vertices of an icosidodecahedron were the edge mid-points of a $2B_1$ icosahedron. From this, it follows that an icosidodecahedron can be constructed by truncating a $2B_1$ icosahedron to its edge mid-points. Truncating the $2B_1$ icosahedron results in removing 12 pyramids, each having a B_1 pentagonal base and B_1 vertical sides.

The volume of the B_1 icosahedron is $\frac{5\phi^2}{6}$; —see section 5.1 on page 18—by applying the scaling rule, the volume of the $2B_1$ icosahedron is $8\frac{5\phi^2}{6}$.

Subtracting 12 copies of the pentagonal pyramid from the $2B_1$ icosahedron gives the volume of the B_1 icosidodecahedron.

7.1 Area of the Unit Pentagon

Consider a unit pentagon of side $B_1 = 1$. Pick any vertex, and construct the $B_2 = \phi$ diagonals incident on that vertex —see figure 1 on page 4. This divides the area of the pentagon into 3 triangles, two of which are congruent —see figure 1 on page 4.

Consider one of these 2 congruent triangles, and observe the $B_1 = 1$ adjacent sides with an included angle of $108°$. The area of this triangle is

$$\frac{1}{2}\sin 108 = \frac{\sin 72}{2}.$$

This gives the area of the 2 congruent triangles to be $\sin 72$.

Next, consider the triangle with sides $\{B_2, B_2, B_1\}$, and observe the adjacent sides of length B_1, B_2 with included angle $72°$. Its area is given by

$$\frac{1}{2}\phi \sin 72.$$

Summing the parts, the area of the unit pentagon is

$$\begin{aligned} A &= \sin 72(1 + \frac{\phi}{2}) \\ &= \frac{(1+\phi)^{\frac{3}{2}}}{4} \qquad \text{(Using (1.6))}. \end{aligned}$$

7.2 Volume of the Pentagonal Pyramid

Consider the center of the pentagon, and observe the triangle formed by connecting it to 2 adjacent vertices of the pentagon. Let r be the radius of the unit pentagon. The central angle is $\frac{360}{5} = 72°$, and the base angles of this isosceles triangle is $\sin 54$. By the sine rule, we have

$$\begin{aligned} \frac{r}{\sin 54} &= \frac{1}{\sin 72} \\ r &= \frac{\sin 54}{\sin 72} \\ &= \frac{1}{2\sin 36}. \end{aligned}$$

Dropping a perpendicular from the apex of the pyramid to its base, and applying the Pythogorian theorem as in section 5.2 on page 20, the height of this pyramid is

$$\begin{aligned} H_P &= \sqrt{(1 - \frac{1}{4\sin^2 36})} \\ &= \sqrt{\frac{4\sin^2 36 - 1}{4\sin^2 36}}, \\ &= \frac{\sqrt{(3 - 4\cos^2 36)}}{2\sin 36} \qquad \text{(Using (1.15))}. \end{aligned}$$

Rewriting $\sin 36$ and $\cos 36$ in terms of ϕ, using equations ((1.4) and (1.15)), and using the identity $\sqrt{(3-\phi^2)} = \frac{1}{\phi}$), we get the height to be

$$\begin{aligned} H_P &= \frac{\sqrt{(3-\phi^2)}}{2} \frac{2\phi}{\sqrt{(1+\phi^2)}} \\ &= \frac{1}{\sqrt{(1+\phi^2)}}. \end{aligned} \tag{7.1}$$

Polyhedron	Side	Vol	Alt	V
Dodecahedron	1	$\phi^3 + 6(\frac{\phi}{4} + \frac{1}{6})$		$2 + \frac{7\phi}{2}$
Icosahedron	1	$\phi^3 - 12\frac{\phi}{24} - 8\frac{\sqrt{5}}{24}$	$\frac{\phi^6}{6} - 12\frac{\phi}{24}$	$5\frac{\phi^2}{6}$
Tetrahedron	$\sqrt{2}$	$\frac{1^3}{3}$	1/3 unit cube volume	
Octahedron	$\sqrt{2}$	$1^3 + \frac{1^3}{3}$	4 times the tetrahedron volume	
Rhombic Dodecahedron	$\sin 60$	$1^3 + 6\frac{1^3}{6}$	Twice unit cube	
Cube-octahedron	$\sqrt{2}$	$\frac{20}{3}$	Chamfered cube	
RT[1]	$\sin 72$	$5\frac{\phi^2}{6}(\phi^3 + \sqrt{2-\phi})$	$\frac{5\phi^3}{2}$	

Table 2: Volumes of regular polyhedra. The volume for each polyhedron is expressed in forms that make the decomposition obvious.

The volume of the pentagonal pyramid is

$$
\begin{aligned}
V_P &= \frac{1}{3}\frac{(1+\phi)^{\frac{3}{2}}}{4}\frac{1}{\sqrt{(1+\phi^2)}} \\
&= \frac{(1+\phi^2)}{12}.
\end{aligned}
$$

Summing the parts.

The The volume of icosidodecahedron is given by

$$
\begin{aligned}
V_{I32} &= \frac{5}{6}\phi^2 8 - (1+\phi^2) \\
&= \frac{17}{3}\phi^2 - 1.
\end{aligned}
$$

8 Conclusion

To conclude, here is a table listing the various formulae derived in this paper.

Finally, here are the 5 platonic solids drawn using package Metapost.

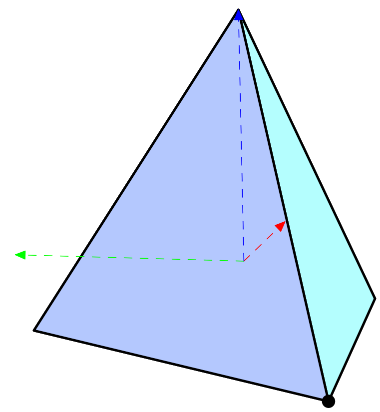

Figure 10: The Tetrahedron has 4 vertices and 4 faces, and is dual to itself.

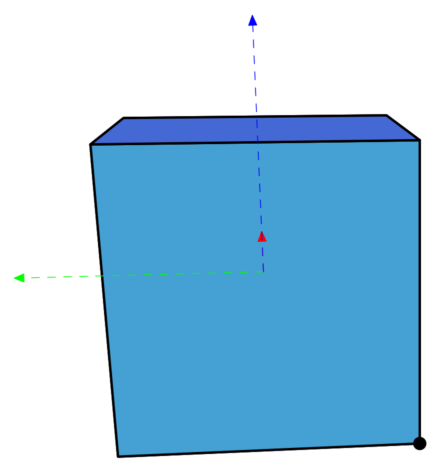

Figure 11: The cube has 8 vertices and 6 faces.

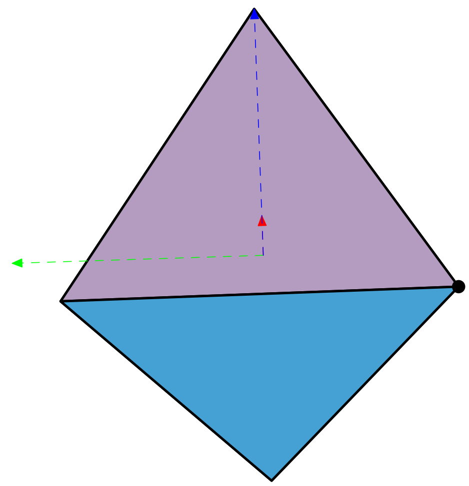

Figure 12: The octahedron has 6 vertices, 8 faces, and is dual to the cube.

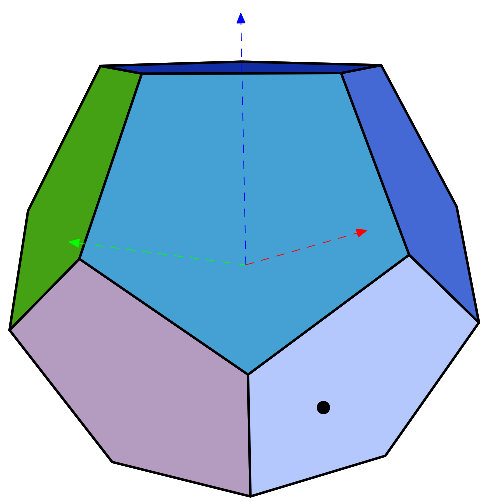

Figure 13: The dodecahedron has 12 faces and 20 vertices.

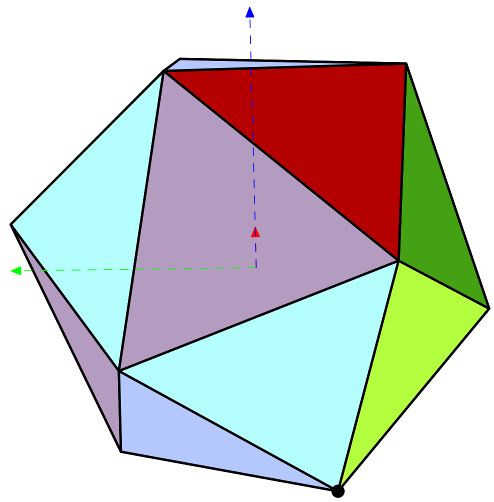

Figure 14: The icosahedron has 12 vertices, 20 faces, and is dual to the dodecahedron.

9 Acknowledgements

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This paper was authored in LaTeX on the Emacspeak audio desktop[2] with Emacs package AucTeX providing authoring support. At a time when most of the attention around Open Source Software is focused on the operating system, we would like to draw readers' attention to the wonderful array of high-quality open source authoring and document preparation tools created over the last 25 years by the (LA)TEX community. All figures in this paper were drawn using declarative authoring packages `pstricks` and `metapost` that enabled the first author to reliably draw these diagrams without having to look at the final output. The high-level markup also makes this content long-lived and reusable —an immediate advantage is that the content can be easily made available using a variety of access modes ranging from high-quality print and online hypertext to high-quality audio renderings[3] see [Ram98]. The document preparation tools used to prepare this paper are well described in [GMS94, GRM97, GR99] and the first author would like to thank author Sabastian Rahtz and their publisher Addison Wesley for providing access to the (LA)TEX

[2] http://emacspeak.sf.net

[3] AsTeR http://www.cs.cornell.edu/home/raman/aster/aster-toplevel.html

sources to these books. We would like to thank author John Hobby for his work on Metapost and acknowledge author Denis Roegel for his metapost macros for drawing three-dimensional polyhedra. Finally, we would like to thank Donald E Knuth for the TEX typesetting system.

References

[Coe73] H. S. M. Coexter. *Regular Polytopes - Third Edition*. Dover, 1973. 1

[Coe89] H. S. M. Coexter. *Introduction to Geometry*. John Wiley and Sons, 1989. 1

[GMS94] Michel Goossens, Frank Mittelbach, and Alexander Samarin. *The LaTeX Companion*. Addison-Wesley, Reading, MA, 1994. 9

[GR99] Michel Goossens and Sebastian Rahtz. *The LaTeX Web Companion —Integrating TeX, HTML, and XML*. Addison-Wesley, Reading, MA, 1999. 9

[GRM97] Michel Goossens, Sebastian Rahtz, and Frank Mittelbach. *The LaTeX Graphics Companion: Illustrating Documents wth TeX and PostScript*. Addison-Wesley, Reading, MA, 1997. 9

[HP01] George Hart and Henri Picciotto. *Zome Geometry: Hands-on Learning with Zome Models*. Key Curriculum Press, 2001. 1.1, 9

[Ram98] T. V. Raman. *AsTeR Audio System For Technical Readings*. Lecture Notes In Computer Science. Springer Verlag, December 1998. 9

[Wei98] Eric W. Weisstein. *The CRC Concise Encyclopedia of Mathematics*. CRC Press, 1998. 1