Lab 21 (2004-12-02)

1. **Substitutions**

   Evaluate the following substitutions based on the two rules of substitution outlined in class:

   - $\{u/x\}(\lambda y.x)$
     This is a substitution of the form $\{N/x\}M$ with $N = u$ and $M = (\lambda y.x)$. The free variable in N i.e. u has no bound occurrences in M, hence substitution is straight forward: replace x with u to get
     $(\lambda y.u)$
   - $\{u/x\}(\lambda u.x)$
     Here the free variable in N has a bound occurrence in M; hence the bound occurrence of u in $(\lambda u.x)$ shall be renamed to a new variable, say z:
     $\{u/x\}(\lambda z.x)$
     Now substitution can be carried out by replacing occurrences of x with u. The answer is
     $(\lambda z.u)$
   - $\{u/x\}(\lambda u.u)$
     Again, the occurrence of u in $M = (\lambda u.u)$ is bound hence rename u in M to say z
     $M = \{u/x\}(\lambda z.z)$
     Now there no occurrences of x in M, hence the replacement of x with u has no effect. The answer is
     $(\lambda z.z)$
   - $\{u/x\}(\lambda y.u)$
     u does occur in $M = (\lambda y.u)$ but it is a free occurrence. Rule 2 (renaming) of substitution applies only for bound occurrences, hence no renaming is required here. Replacing x with u has no effect either, since M does not contain any occurrences of x. Hence the substituted result is
     $(\lambda y.u)$

2. **Reduction Sequences**

   This question is for practise on reduction sequences. It should help you to check your understanding of reduction sequences, as well as the syntax and strategies involved.

   - Using the α and β reductions discussed in the lecture, reduce the following expression:

     $T = (\lambda xyz.xz(yz))(\lambda x.x)(\lambda x.x)$

     The first step should be to identify *redexes* i.e. terms of the form $(\lambda x.M)N$. Remember that in the presence of several redexes in an expression, a reduction strategy guides you on which one to pick first. Apply both *call-by-name* and *call-by-value* strategies here and verify that the result is the same, hence upholding the Church-Rosser theorem.

There is only one redex in T, upon β-reduction we get
$T' = (\lambda yz.(\lambda x.x)z(yz))(\lambda x.x)$
This term has two redexes now, the entire term is a redex and so is the subterm $(\lambda x.x)z$.
Hereon, Call-by-value proceeds as follows (the redex being reduced at each step is underlined):

$$\begin{aligned} T' = (\lambda yz.\underline{(\lambda x.x)z}(yz))(\lambda x.x) &\Rightarrow_\beta \underline{(\lambda yz.z(yz))(\lambda x.x)} \\ &\Rightarrow_\beta \lambda z.z(\underline{(\lambda x.x)z}) \\ &\Rightarrow_\beta \lambda z.zz \end{aligned}$$

Call-by-name achieves the same result:

$$\begin{aligned} T' = \underline{(\lambda yz.(\lambda x.x)z(yz))(\lambda x.x)} &\Rightarrow_\beta (\lambda z.\underline{(\lambda x.x)z}(\lambda x.xz)) \\ &\Rightarrow_\beta \lambda z.z(\underline{(\lambda x.x)z}) \\ &\Rightarrow_\beta \lambda z.zz \end{aligned}$$

- An interesting case is the reduction of (Yf) where

  $Y = (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))$

  A term Y of this form is known as a *fixed-point combinator* and is used to set up recursions in λ calculus. A detailed discussion on fixed-point combinators is beyond the scope of this exercise, nevertheless the reduction of (Yf) should help illustrate the concept to some extent. Prove the result
  $Yf = f(Yf)$

  $$\begin{aligned} Yf &= (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))f \\ &\Rightarrow_\beta (\lambda x.f(xx))(\lambda x.f(xx)) \\ &\Rightarrow_\beta f((\lambda x.f(xx))(\lambda x.f(xx))) \end{aligned}$$

  The last expression can be rewritten (based on its previous one) to
  $= f(Yf)$