Foundations of Computer Science— CSCI 2200
Exam 2

Question 1 (6 points). Suppose you accept a job at Odd Corporation with the following terms. You start
the job on January 1, 2020, and your initial salary is $100,000. On January 1 of every odd numbered year (e.g.,
2021, 2023, 2025, ...), you receive a 5% raise. Give a recursive definition for your salary in your $n^{th}$ year of
employment. You do not need to show your work.

Solution 1:

Basis step: $a_1 = 100000$
Recursive step: $a_n = (1.05)^{(n+1) \mod 2} a_{n-1}$, for $n > 1$

Solution 2:

Basis step: $a_1 = 100000$, $a_2 = 100000 \cdot 1.05$
Recursive step $a_n = 1.05a_{n-2}$, for $n > 2$

Question 2 (6 points). Consider the alphabet $\Sigma = \{X,Y\}$. Let $S$ be the set of all strings over the alphabet
$\Sigma$ in which the symbols appear in lexicographical order, meaning strings in which no $Y$ appears before any $X$.
The empty string is in $S$. Give a recursive definition for $S$.

Solution:

Basis step: $\lambda \in S$
Recursive step: if $w \in S$ then $Xw \in S$ and $wY \in S.$
Question 3 (12 points). Prove by mathematical induction that for any positive integer $n$, $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Solution: This is a proof by induction.

Basis step: $(n = 1)$ $1^3 + (1 + 1)^3 + (1 + 2)^3 = 1 + 8 + 27 = 36$
$36 = 4 \cdot 9$, so 36 is divisible by 9.

Inductive step: Assume $k^3 + (k + 1)^3 + (k + 2)^3$ is divisible by 9, for some positive integer $k$. We will show this implies $(k + 1)^3 + (k + 2)^3 + (k + 3)^3$ is divisible by 9.

First note that

$$(k + 1)^3 + (k + 2)^3 + (k + 3)^3 = (k + 1)^3 + (k + 2)^3 + (k^3 + 9k^2 + 27k + 27)
= k^3 + (k + 1)^3 + (k + 2)^3 + 9k^2 + 27k + 27
= k^3 + (k + 1)^3 + (k + 2)^3 + 9(k^2 + 3k + 3).$$

By the inductive hypothesis, $k^3 + (k + 1)^3 + (k + 2)^3$ is divisible by 9. Therefore, $k^3 + (k + 1)^3 + (k + 2)^3 = 9j$ for some integer $j$. The sum above can then be written as:

$$9j + 9(k^2 + 3k + 3) = 9(k^2 + 3k + 3 + j).$$

Since $k^2 + 3k + 3 + j$ is an integer, the sum $(k + 1)^3 + (k + 2)^3 + (k + 3)^3$ is divisible by 9. QED
**Question 4 (12 points).** Let \( a_n \) be the sequence defined by \( a_1 = 1, a_2 = 8 \), and \( a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 3 \).

Prove by strong induction that \( a_n = 3 \cdot 2^{n-1} + 2(-1)^n \) for all positive integers \( n \).

_Hint: note that \((-1)^{r-1} = (-1)^2(-1)^{r-1} = (-1)^{r+1} \) for all integers \( r > 1 \)._

**Solution:** This is a proof by strong induction.

Basis step:
\[
(n = 1) \quad 3 \cdot 2^1 - 1 + 2(-1)^1 = 3 \cdot 1 - 2 = 1 \\
(n = 2) \quad 3 \cdot 2^2 - 1 + 2(-1)^2 = 3 \cdot 2 + 2 = 8
\]

Inductive step:
Assume: \( a_j = 3 \cdot 2^{j-1} + 2(-1)^j \), for \( 1 \leq j \leq k \), with \( k \geq 2 \).
We will show this implies \( a_{k+1} = 3 \cdot 2^k + 2(-1)^{k+1} \).

By the recursive definition of \( a_n \),
\[
a_{k+1} = a_k + 2a_{k-1}.
\]

Applying the inductive hypothesis, we obtain:
\[
a_{k+1} = 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{(k-1)-1} + 2(-1)^{k-1}) \\
= 3 \cdot 2^{k-1} + 3 \cdot 2^{k-1} + 2(-1)^k + 2 \cdot 2(-1)^{k-1} \\
= 3 \cdot 2^k + 2(-1)^k + 2 \cdot 2(-1)^{k-1} \\
= 3 \cdot 2^k + 2(-1)(-1)^k + 2 \cdot 2(-1)^{k-1} \\
= 3 \cdot 2^k + 2(-1) \cdot (-1)^{k-1} + 2 \cdot 2(-1)^{k-1} \\
= 3 \cdot 2^k + 2(-1)^{k-1} + 2 \cdot 2(-1)^{k-1} \\
= 3 \cdot 2^k + 2(-1)^k + 2(-1)^{k+1},
\]
where the last equality follows from the hint. QED
Question 5 (2 + 2 + 10 points). Recall the recursive definition of a full binary tree:

Basis step: A single vertex $r$ is a full binary tree.

Recursive step: Suppose that $T_1$ and $T_2$ are disjoint full binary trees. Then there is a tree $T = T_1 \circ T_2$ consisting of a root $r$ with edges connecting to the roots of $T_1$ and $T_2$.

(a) Give a recursive definition of the number of vertices $V(T)$ in the full binary tree $T$.

Solution:
Basis step: For a tree $T$ consisting of a single node, $V(T) = 1$

Inductive step: For $T = T_1 \circ T_2$, where $T_1$ and $T_2$ are disjoint full binary trees: $V(T) = V(T_1) + V(T_2) + 1$

(b) Give a recursive definition of the number of edges $E(T)$ in the full binary tree $T$.

Solution:
Basis step: For a tree $T$ consisting of a single node, $E(T) = 0$

Inductive step: For $T = T_1 \circ T_2$, where $T_1$ and $T_2$ are disjoint full binary trees: $E(T) = E(T_1) + E(T_2) + 2$

(c) Give a proof by structural induction that for all full binary trees $T$, $V(T) = E(T) + 1$.

Solution: This is a proof by structural induction.

Basis step: For a tree $T$ consisting of a single node, $V(T) = 1$ and $E(T) = 0$, so $V(T) = E(T) + 1$.

Inductive step: Assume that for disjoint full binary trees $T_1$ and $T_2$,

$V(T_1) = E(T_1) + 1$
$V(T_2) = E(T_2) + 1$

We will show this implies that for $T = T_1 \circ T_2$, $V(T) = E(T) + 1$.

First, note that by the recursive definition of $V$, $V(T) = V(T_1) + V(T_2) + 1$. Applying the inductive hypothesis, we obtain

$$V(T) = E(T_1) + 1 + E(T_2) + 1 + 1$$
$$= E(T_1) + E(T_2) + 2 + 1$$
$$= E(T) + 1,$$

where the last equality follows from the recursive definition of $E(T)$. QED