Foundations of Computer Science— CSCI 2200
Exam 2

Question 1 (6 points). Suppose you accept a job at Odd Corporation with the following terms. You start the job on January 1, 2020, and your initial salary is \$100,000. On January 1 of every odd numbered year (e.g., 2021, 2023, 2025, ...), you receive a 5% raise. Give a recursive definition for your salary in your n^{th} year of employment. You do not need to show your work.

Solution 1:

Basis step: $a_1 = 100000$
Recursive step: $a_n = (1.05)^{(n+1) \bmod 2} a_{n-1}$, for $n > 1$

Solution 2:

Basis step: $a_1 = 100000$, $a_2 = 100000 \cdot (1.05)$
Recursive step $a_n = 1.05 a_{n-2}$, for $n > 2$

Question 2 (6 points). Consider the alphabet $\Sigma = \{X, Y\}$. Let S be the set of all strings over the alphabet Σ in which the symbols appear in lexicographical order, meaning strings in which no Y appears before any X. The empty string is in S. Give a recursive definition for S.

Solution:
Basis step: $\lambda \in S$
Recursive step: if $w \in S$ then $Xw \in S$ and $wY \in S$.

Question 3 (12 points). Prove by mathematical induction that for any positive integer n, $n^3+(n+1)^3+(n+2)^3$ is divisible by 9.

Solution: This is a proof by induction.

Basis step: $(n = 1)$ $1^3 + (1 + 1)^3 + (1 + 2)^3 = 1 + 8 + 27 = 36$
$36 = 4 \cdot 9$, so 36 is divisible by 9.

Inductive step: Assume $k^3 + (k + 1)^3 + (k + 2)^3$ is divisible by 9, for some positive integer k.
We will show this implies $(k + 1)^3 + (k + 2)^3 + (k + 3)^3$ is divisible by 9.

First note that

$$\begin{aligned}(k+1)^3 + (k+2)^3 + (k+3)^3 &= (k+1)^3 + (k+2)^3 + (k^3 + 9k^2 + 27k + 27) \\ &= k^3 + (k+1)^3 + (k+2)^3 + 9k^2 + 27k + 27 \\ &= k^3 + (k+1)^3 + (k+2)^3 + 9(k^2 + 3k + 3).\end{aligned}$$

By the inductive hypothesis, $k^3 + (k + 1)^3 + (k + 2)^3$ is divisible by 9. Therefore, $k^3 + (k + 1)^3 + (k + 2)^3 = 9j$ for some integer j. The sum above can then be written as:

$$9j + 9(k^2 + 3k + 3) = 9(k^2 + 3k + 3 + j).$$

Since $k^2 + 3k + 3 + j$ is an integer, the sum $(k + 1)^3 + (k + 2)^3 + (k + 3)^3$ is divisible by 9. QED

Question 4 (12 points). Let a_n be the sequence defined by $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove by strong induction that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all positive integers n.
Hint: note that $(-1)^{r-1} = (-1)^2(-1)^{r-1} = (-1)^{r+1}$ *for all integers* $r > 1$.

Solution: This is a proof by strong induction.

Basis step:
$(n = 1)$ $3 \cdot 2^{1-1} + 2(-1)^1 = 3 \cdot 1 - 2 = 1$
$(n = 2)$ $3 \cdot 2^{2-1} + 2(-1)^2 = 3 \cdot 2 + 2 = 8$

Inductive step:
Assume: $a_j = 3 \cdot 2^{j-1} + 2(-1)^j$, for $1 \leq j \leq k$, with $k \geq 2$.
We will show this implies $a_{k+1} = 3 \cdot 2^k + 2(-1)^{k+1}$.

By the recursive definition of a_n,

$$a_{k+1} = a_k + 2a_{k-1}.$$

Applying the inductive hypothesis, we obtain:

$$\begin{aligned}
a_{k+1} &= 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{(k-1)-1} + 2(-1)^{k-1}) \\
&= 3 \cdot 2^{k-1} + 3 \cdot 2^{k-1} + 2(-1)^k + 2 \cdot 2(-1)^{k-1} \\
&= 3 \cdot 2^k + 2(-1)^k + 2 \cdot 2(-1)^{k-1} \\
&= 3 \cdot 2^k + 2(-1)(-1)^{k-1} + 2 \cdot 2(-1)^{k-1} \\
&= 3 \cdot 2^k + 2(-1)^{k-1} \\
&= 3 \cdot 2^k + 2(-1)^{k+1},
\end{aligned}$$

where the last equality follows from the hint. QED

Question 5 (2 + 2 + 10 points). Recall the recursive definition of a *full binary tree*:
Basis step: A single vertex r is a full binary tree.
Recursive step: Suppose that T_1 and T_2 are disjoint full binary trees. Then there is a tree $T = T_1 \circ T_2$ consisting of a root r with edges connecting to the roots of T_1 and T_2.

(a) Give a recursive definition of the number of vertices $V(T)$ in the full binary tree T.

Solution:
Basis step: For a tree T consisting of a single node, $V(T) = 1$

Inductive step: For $T = T_1 \circ T_2$, where T_1 and T_2 are disjoint full binary trees: $V(T) = V(T_1) + V(T_2) + 1$

(b) Give a recursive definition of the number of edges $E(T)$ in the full binary tree T.

Solution:
Basis step: For a tree T consisting of a single node, $E(T) = 0$

Inductive step: For $T = T_1 \circ T_2$, where T_1 and T_2 are disjoint full binary trees: $E(T) = E(T_1) + E(T_2) + 2$

(c) Give a proof by structural induction that for all full binary trees T, $V(T) = E(T) + 1$.

Solution: This is a proof by structural induction.

Basis step: For a tree T consisting of a single node, $V(T) = 1$ and $E(T) = 0$, so $V(T) = E(T) + 1$.

Inductive step: Assume that for disjoint full binary trees T_1 and T_2,
$V(T_1) = E(T_1) + 1$
$V(T_2) = E(T_2) + 1$
We will show this implies that for $T = T_1 \circ T_2$, $V(T) = E(T) + 1$.

First, note that by the recursive definition of V, $V(T) = V(T_1) + V(T_2) + 1$. Applying the inductive hypothesis, we obtain

$$\begin{aligned} V(T) &= E(T_1) + 1 + E(T_2) + 1 + 1 \\ &= E(T_1) + E(T_2) + 2 + 1 \\ &= E(T) + 1, \end{aligned}$$

where the last equality follows from the recursive definition of $E(T)$. QED