- Intractability and the Hierarchy Theorems – Sipser 9.1

Some material from slides by M. Sipser
Announcements

- Homework 5 is posted in Submitty Course Materials.
  - It is due on April 23, 2021 in Gradescope.
What we’ve learned so far:

\[ L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE = NPSPACE \subseteq EXPTIME \]

We do not know whether any of these subset relations (\(\subseteq\)) are strict (\(\subset\)).

Intuitively, giving a TM more time or space should increase the class of problems it can solve.

The hierarchy theorems prove this intuition is correct

- The theorems don’t tell us specifically whether \(P \subseteq NP\), for example, but they do give us more information about the relationship between complexity classes.
Review – Asymptotic Notation

- **Big-O Notation**
  - Let $f$ and $g$ be functions, $f, g: \mathbb{N} \to \mathbb{R}^+$.  
  - We say $f(n) = O(g(n))$ if there exists positive integers $c$ and $n_0$ such that every integer $n \geq n_0$, $f(n) \leq c \cdot g(n)$.  
  - $g(n)$ is an **asymptotic upper bound** for $f(n)$.

- **Little-o Notation**
  - Let $f$ and $g$ be functions, $f, g: \mathbb{N} \to \mathbb{R}^+$.  
  - We say $f(n) = o(g(n))$ if for every $c > 0$ there exists an $n_0$ such that $f(n) < c \cdot g(n)$ for every integer $n \geq n_0$.

- The difference between big-O and little-o is analogous to the difference between $\leq$ and $<$.  

Space Hierarchy Theorem

- **Theorem:** For any computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ (that satisfies certain technical properties), there exists a language $A$ that is decidable in $O(f(n))$ space but not in $o(f(n))$ space.
D: on input $w$:

1. Let $n = |w|$.
2. Compute $f(n)$ and mark off this much tape.
   If later stages use more than $f(n)$ tape, reject.
3. If $w$ is not of form $<M>10^*$ for some TM $M$, reject.
4. Simulate $M$ on $w$ while counting the number of steps used.
   If count exceeds $2^{f(n)}$, reject.
5. If $M$ accepts, reject. If $M$ rejects, accept.
D: on input $w$:

1. Let $n = |w|$.
2. Compute $f(n)$ and mark off this much tape.
   If later stages use more than $f(n)$ tape, reject.
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Space Constructable Function

- **Theorem:** For any computable function $f: \mathbb{N} \to \mathbb{N}$ (that satisfies certain technical properties), there exists a language $A$ that is decidable in $O(f(n))$ space but not in $o(f(n))$ space.

- A function $f: \mathbb{N} \to \mathbb{N}$ where $f(n)$ is at least $O(\log n)$ is **space constructable** if the function $h(1^n) = f(n)$ (in binary) is computable in space $O(f(n))$.

- **Theorem:** For any space constructable function $f: \mathbb{N} \to \mathbb{N}$, there exists a language $A$ that is decidable in $O(f(n))$ space but not in $o(f(n))$ space.
Implications of the Space Hierarchy Theorem

- **Theorem:** For any space constructable function \( f : \mathbb{N} \rightarrow \mathbb{N} \), there exists a language \( A \) that is decidable in \( O(f(n)) \) space but not in \( o(f(n)) \) space.

- **Corollary:** \( NL \subseteq PSPACE \)

- **Corollary:** \( PSPACE \subseteq EXPSPACE \)
The Time Hierarchy Theorem

- **Theorem:** For any $f: \mathbb{N} \to \mathbb{N}$ where $f$ is time constructible there is a language $A$ where
  
  1) $A$ is decidable in $O(f(n))$ time, and
  
  2) $A$ is not decidable in $o\left(\frac{f(n)}{\log(f(n))}\right)$ time.
D: on input $w$:
1. Let $n = |w|$
2. Compute $f(n)$ and store value in a binary counter.
   Decrement counter before each step in stages 4 and 5. If counter hits 0, reject.
3. If $w$ is not of form $\langle M \rangle 01^*$ for some TM $M$, reject.
4. Simulate $M$ on $w$.
5. If $M$ accepts, then reject. If $M$ rejects, then accept.
D: on input $w$:

1. Let $n = |w|$
2. Compute $f(n)$ and store value in a binary counter. Decrement counter before each step in stages 4 and 5. If counter hits 0, reject.
3. If $w$ is not of form $⟨M⟩01^∗$ for some TM $M$, reject.
4. Simulate $M$ on $w$.
5. If $M$ accepts, then reject. If $M$ rejects, then accept.
Implications of the Time Hierarchy Theorem

- **Time Hierarchy Theorem**: For any $f: \mathbb{N} \rightarrow \mathbb{N}$ where $f$ is time constructible there is a language $A$ where
  1) $A$ is decidable in $O(f(n))$ time, and
  2) $A$ is not decidable in $o(f(n) / \log(f(n)))$ time

- **Corollary**: $P \subset \text{EXPTIME}$
Example

- Prove that $NTIME(n) \subseteq PSPACE$. 