- Mapping Reducibility (Section 5.3)
- The Post Correspondence Problem (Section 5.2)
Announcements

- No class on Thursday, April 1, 2021.
For two languages (problems) A and B, A is **reducible** to B if we can use a solution to B to solve A.

- If A is reducible to B:
  - If B is decidable, then A is decidable.
  - If A is undecidable, then B is decidable.

We used the above to show several problems were undecidable.

- In these examples, \( A = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \} \)
  - We showed A was undecidable using a different method.
- For the halting problem: \( B = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \} \)
- For the empty language problem: \( B = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \} \)
Quick Recap – The Halting Problem

- To prove $B = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ halts on input } w \}$ is undecidable.

- Assume, for contradiction, that $B$ is decidable. Let $R$ be a decider for $B$.

- We construct a decider $S$ for $A = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \}$.

- $S$ - On input $\langle M, w \rangle$ where $M$ is a Turing machine and $w$ is a string:
  - Run $R$ on $\langle M, w \rangle$
  - If $R$ rejects, output reject
  - If $R$ accepts
    - Simulate $M$ on $w$ until it halts
    - If $M$ accepts, output accept
    - If $M$ rejects, output reject
We used the reducibility technique to prove the halting problem is undecidable.

Now, we formalize this technique.

We can also use this formalization for to show certain languages are not Turing-recognizable and for applications in complexity theory.

A function \( f : \Sigma^* \rightarrow \Sigma^* \) is a **computable function** if some Turing machine \( M \), on every input \( w \), halts with just \( f(w) \) on its tape.

Language \( A \) is **mapping reducible** to language \( B \) (\( A \leq_m B \)) if there is a computable function \( f : \Sigma^* \rightarrow \Sigma^* \) (the **mapping reduction**) where \( w \in A \) if and only if \( f(w) \in B \).
Theorem: if $A \leq_m B$ and $B$ is decidable, then $A$ is decidable.
The language $B = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \}$ is undecidable.
Theorem: if $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.

Corollary: if $A \leq_m B$ and $A$ is not Turing-recognizable, then $B$ is not Turing-recognizable.

Useful example of language that is not Turing-recognizable:

- $A = \{ \langle M, w \rangle \mid M$ is a Turing machine and $M$ accepts $w \}$ is Turing-recognizable.
- $\overline{A}$ is not Turing-recognizable.
The Post Correspondence Problem

SECTION 5.2
The Post Correspondence Problem (PCP)

Given a collection of pairs of strings as dominos:

\[ P = \left\{ \begin{bmatrix} t_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} t_2 \\ b_2 \end{bmatrix}, \ldots, \begin{bmatrix} t_k \\ b_k \end{bmatrix} \right\} \]

a match is a finite sequence of dominos in \( P \) (repeats allowed) where the concatenation of the \( t \)'s = the concatenation of the \( b \)'s.

\[ \text{Match} = \begin{bmatrix} t_{i_1} \\ b_{i_1} \end{bmatrix} \begin{bmatrix} t_{i_2} \\ b_{i_2} \end{bmatrix} \ldots \begin{bmatrix} t_{i_l} \\ b_{i_l} \end{bmatrix} \]

where \( t_{i_1} t_{i_2} \ldots t_{i_l} = b_{i_1} b_{i_2} \ldots b_{i_l} \)
Example PCP

- Does P have a match?

$$P = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 011 \\ 01 \end{bmatrix}, \begin{bmatrix} 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0001 \end{bmatrix}, \begin{bmatrix} 00 \\ 0 \end{bmatrix}, \begin{bmatrix} 100 \\ 00 \end{bmatrix} \}$$
PCP = \{ \langle P \rangle \mid P \text{ is a } \text{-instance of the Post Correspondence Problem and } P \text{ has a match} \}

**Theorem:** PCP is undecidable.
Modified Post Correspondence Problem

- MPCP = \{ \langle P \rangle | P \text{ is an instance of the Post Correspondence Problem with a match that starts with the first domino} \}
- A = \{ \langle M, w \rangle | M \text{ is a Turing machine and } M \text{ accepts } w \} \text{ is Turing-recognizable.}

**Theorem:** A \leq_m MP CP