1 Matching

A **matching** M in a graph G is a set of non-loop edges with no shared endpoints. Vertices incident to M are **saturated**; vertices not incident to M are **unsaturated**. A **perfect matching** is a matching that saturates all $v \in V(G)$. A **maximal matching** is a matching that can't be extended with the addition of an edge. A **maximum matching** is a matching that is the maximum size over all possible matchings on G.

Given a matching M, an M**-alternating path** is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoint vertices are unsaturated by M is an M**-augmenting path**. **Berge's Theorem** states that a matching M of G is a maximum matching if and only if G has no M-augmenting path.

The **symmetric difference** between two graphs G and H, written as $G\Delta H$, is the subgraph of $G \cup H$ whose edges are the edges that appear in only one of G and H. The symmetric difference between two matchings contains either paths or cycles.

Hall's Theorem states that An X, Y-bipartite graph G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all possible $S \subseteq X$. **Hall's Condition** implies $\forall S \subseteq X, |N(S)| \geq |S|$ for X to be saturated. We can therefore show that a bigraph has no matching saturating X if we identify a subset $S \subseteq X$ where $|N(S)| < |S|$.

We can use Hall's theorem to show that all k-regular bipartite graphs have a perfect matching.

2 Independent Sets and Covers

A **vertex cover** of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint on all $e \in E(G)$. The vertices in Q *cover* $E(G)$. An **edge cover** of G is a set $L \subseteq E(G)$ such that L has at least one edge incident on all $v \in V(G)$. The edges in L *cover* $V(G)$.

The **König-Egerváry Theorem** states that if G is a bipartite graph, then the size of a maximum matching in G equals the minimum size of a vertex cover.

As we've previously discussed, an **independent set** of vertices on a graph G are a set of vertices that are not connected by an edge. The size of a maximum independent set on G is called the **independence number** of G. For a bipartite graph, this isn't necessarily the size of the larger partite set.

In G, $S \subseteq V(G)$ is an independent set if and only if $\bar{S}$ is a vertex cover. Thus a maximum independent set is the complement of a minimum vertex cover, and their sizes summed equals the order of G.

3 Maximum Bipartite Matching

In unweighted bipartite graphs, we can iteratively increase the size of an initial matching M by finding augmenting paths. If an augmenting path can't be found, we know via **Berge's Theorem** that we have a maximum match. The **Augmenting Path Algorithm** is below. For unweighted shortest paths, we can simply use breadth-first search as talked about previously.

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procedure MATCHBIPARTITE(X, Y-bigraph G)
  $M \leftarrow \emptyset$ ▷ M initially empty
  do
    $P \leftarrow$ AugPathAlg(G, M) ▷ New augmented path found with M, G
    $M \leftarrow M \Delta P$ ▷ Symmetric difference between M, P
  while $P \neq \emptyset$
  return M

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procedure AUGPATHALG(X, Y-bigraph G and matching $M = (V_M, E_M)$)
  $G' \leftarrow G$
  Orient $G' : \forall e \in E_M : e(x_i, y_j) = e(y_j \to x_i); \forall e \notin E_M : e(x_i, y_j) = e(x_i \to y_j)$
  Add vertex s to G' with edges $\forall x_i \in X, x_i \notin V_M : (s \to x_i)$
  Add vertex t to G' with edges $\forall y_j \in Y, y_j \notin V_M : (y_j \to t)$
  $P \leftarrow$ ShortestPathBFS(G', s, t) ▷ Use BFS to find shortest path from s to t
  return $P - \{e(s, x_i), e(y_j, t)\}$ ▷ Return path without added edges

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