1   Vertex Coloring

Assume all graphs are connected and without self loops.

A k-**coloring** of a graph is a labeling $f : V(G) \to S$, with $k = |S|$. Essentially, each vertex is assigned a **color** of $1 \ldots k$. A coloring is **proper** if no adjacent vertices share the same color. A graph is k-**colorable** if it has a proper k-coloring.

The **chromatic number** of a graph $\chi(G)$ is the least k for which G is k-colorable. A graph is k-**chromatic** if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an **optimal coloring**. Note that an optimal coloring is not necessarily unique.

If $\chi(H) < \chi(G) = k$ for every subgraph H of G, then G is **color-critical**.

Remember back to when we discussed independent sets. We can observe that sets comprised of vertices of the same color in a proper coloring are all independent.

2   Greedy Coloring

A simple greedy algorithm for creating a proper coloring is shown below. The basic idea is do a single pass through all vertices of the graph in some order and label each one with a numeric identifier. A vertex will labeled/colored with the lowest value that doesn't appear among previously colored neighbors.

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procedure GREEDYCOLORING(Graph $G(V, E)$)
    for all $v \in V(G)$ **do**
        $color(v) \leftarrow -1$
    for all $v \in V(G)$ in order **do**
        $isCol(1 \ldots \Delta(G) + 1) \leftarrow$ **false**
        for all $u \in N(v)$ where $color(u) \neq -1$ **do**
            $isCol(color(u)) \leftarrow$ **true**
        for $k = 1 \ldots \Delta(G) + 1$ **do**
            if $isCol(k) =$ **false then**
                $color(v) \leftarrow k$
                break
    $k \leftarrow \max(color(1 \ldots n))$
    return k

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3   Coloring Bounds

Obviously, $1 \leq \chi(G) \leq n$.

The clique number of G, $\omega(G)$, is the maximum size of any clique in G. We can see that $\chi(G) \geq \omega(G)$.

Remember that the independence number, $\alpha(G)$, is the size of the largest possible independent set. We can also see that $\chi(G) \geq \frac{n}{\alpha(G)}$.

If $\Delta(G)$ is the maximal degree in a graph, then use of our greedy coloring algorithm shows that $\chi(G) \leq \Delta(G) + 1$.

Running our greedy coloring algorithm on a selected vertex order can slightly improve our bounds. If we run on vertices in non-increasing order, or $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\chi(G) \leq 1 + \max_i, \min\{d_i, i - 1\}$. This gives a bound of one plus the maximum over all vertices of the lesser of either the vertices' degree or its index in the ordering minus one. Note that this bound will be equivalent to the one above should there be more than $\Delta(G)$ vertices with a degree of $\Delta(G)$.

However, we can improve this bound slightly further by creating an even smarter vertex order. If we are able to create a vertex order such that each vertex has at most $k - 1 : k = \Delta(G)$ lower-indexed neighbors, then greedy coloring will give us a bound $\chi(G) \leq \Delta(G)$. **Brooks' Theorem** states we can create such an order, i.e. have the given bound, for all graphs except cliques and cycle graphs with an odd length.