1 More on Coloring Bounds

We've noted before that $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the size of the largest clique in G. When $\chi(H) = \omega(H)$ for all subgraphs H in G, we would say that the graph is **perfect**. However, in practice, the above bound is very loose. Triangle-free graphs can have an arbitrarily large chromatic number.

One way to create triangle-free graphs is through **Mycielski's Construction**. Given triangle-free graph G with vertices $\{v_1, \ldots, v_n\}$ and $\chi(G) = k$, we can create G' with $\chi(G') = k + 1$ by adding vertices $\{u_1, \ldots, u_n\}$ and vertex w to G, making u_i adjacent to all in $N(v_i)$ and having $N(w) = \{u_1, \ldots, u_n\}$.

We can prove that this construction produces a triangle-free graph with a larger chromatic number.

2 Extreme Coloring

We'll now examine the structure of k-chromatic graphs.

How small can a k-chromatic graph be? We can show that every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges.

How large can a k-chromatic (simple) graph be? Let's think in terms of multipartite graph. A **complete multipartite graph** is a generalization of a complete bipartite graph for an arbitrarily large number of independent sets, where $\forall u, v \in V(G) : (u, v) \in E(G)$ if any only if u and v are in different sets. Obviously, a complete multipartite graph is k-chromatic when there are k sets. A **Turán Graph** is the complete r-partite graph with n vertices whose partite sets differ in size by at most 1, i.e., they have sizes of either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

Among simple r-partite graphs with n vertices, the Turán graph is the unique graph with the most edges. Further, among n vertex graphs with no $r + 1$-clique, the Turán graph has the maximum number of edges.

3 Color-critical Graph

A graph G is **color-critical** when every subgraph H of G has a lesser chromatic number. This implies that the removal of any edge or vertex from G decreases the minimal number of colors required for a proper coloring, or $\chi(G - e) < \chi(G)$. To show a graph is color-critical, we only need to compare it with subgraphs obtained by removing a single edge.

For a k-critical graph, for all $v \in V(G)$ where on a proper k-coloring of G the color on v appears nowhere else and the other $k-1$ colors appears in $N(V)$. Additionally, for all $e \in E(G)$, every proper $k-1$ coloring of $G-e$ gives the same color to the two endpoints of e.

If G is a graph with $\chi(G) > k$ and has partitions X, Y, where $G[X]$ and $G[Y]$ are k-colorable, then the edge cut $[X, Y]$ has at least k edges.

Every k-critical graph is $k-1$-edge-connected.