1 One More Bound

Before we start on counting possible colorings, we'll prove one more bound of $\chi(G)$. If $\{G_i\}_{i=1}^{b}$ are the b biconnected components of G, then $\chi(G) = \max_i \chi(G_i)$.

2 Counting Colorings

Although we can't explicitly determine (easily) if a graph has a proper k-coloring, we are able to implicitly approach the problem through examining the number of colorings for fixed k values. Given a Graph G and number of colors k, $\chi(G;k)$ is the number of proper colorings of G. If $\chi(G) < k$, then it is possible to create multiple colorings by changing the colors used on a proper coloring.

Consider clique K_n. The value of $\chi(K_n, k) = k(k-1)\cdots(k-n+1)$

If T is a tree with n vertices, then $\chi(T;k) = k(k-1)^{n-1}$.

We can recognize the fact that each k-coloring of G partitions G into k independent sets. Grouping the colorings according to this partition leads to a formula for $\chi(G;k)$ that is polynomial in k of degree n. $\chi(G;k)$ as a function of k is called the **chromatic polynomial**.

Let $x_{(r)} = x(x-1)\cdots(x-r+1)$. If $p_r(G)$ denotes the number of partitions of $V(G)$ into r nonempty independent sets, then $\chi(G;k) = \sum_{r=1}^{n} p_r(G)k_{(r)}$, which is polynomial in degree n. Determining where this polynomial intercepts the x-axis would help us determine the chromatic number of the graph we are computing it on.

However, directly computing the chromatic polynomial is generally infeasible. But, similar to how we counted spanning trees in G, there is a recurrence relation that we can use. If G is a simple graph and $e \in E(G)$, then $\chi(G;k) = \chi(G-e;k) - \chi(G \cdot e;k)$.

3 Chordal Graphs

A vertex is **simplicial** if its neighborhood in G is a clique. A **simplicial elimination ordering** is an ordering of vertices $\{v_n, \ldots, v_1\}$ for deletion such that each vertex v_i is simplicial in the remaining graph induced by $\{v_n, \ldots, v_i\}$.

A **chord** of a cycle C is an edge not in C but adjacent to two vertices in C. A **chordless cycle** in G is a cycle of at least length 4 that has no chord. A graph G is **chordal** if it is simple and has no chordless cycle.

A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.

Remember that a graph G is **perfect** if $\chi(H) = \omega(H)$, where $\omega(H)$ is the size of the largest clique in H, for all induced subgraphs H of G. We can use the notion of simplicial elimination orderings to prove that chordal graphs are perfect.