1 Coloring of Planar Graphs

One of the more well-studied problems related to coloring and graph planarity is the question of how many colors are required in order to color a map such that any two neighboring regions have different colors. As a map can be represented as a planar graph, where each region is given by a vertex and neighbor relations are given by edges between vertices, the problem is equivalent to asking what is the maximum chromatic number of a planar graph.

We can use Euler's formula and the concept of **Kempe Chains**, paths in which there are two colors that alternate, to show that every planar graph is 5-colorable. This is the **Five Color Theorem**.

However, is this the lower bound on the chromatic number of a planar graphs? Let's try and reduce our bound to four colors. An approach for the proof involves us seeking a minimal counterexample, which is an unavoidable set of graphs that can't be present in a four-colored planar graph. We consider triangulations, since every simple planar graph can be contained in a triangulation.

A **configuration** in a planar triangulation is a separating cycle C with the portions of G inside of C. A configuration is **unavoidable** if a minimal counterexample must contain a member of it. A configuration is **reducible** if a planar graph containing it cannot be a minimal counterexample.

Since the minimal degree of a planar graph is less than or equal to five and in a triangulation the minimum degree is greater than or equal to three, then we immediately have three unavoidable configurations. For these, C is length three, four and five each with single vertex in the middle attached to each vertex in C.

We can try and prove that planar graphs are four-colorable using these configurations. However, the configuration using a cycle of length five isn't shown as reducible, so larger configurations must be considered.

The **Four Color Theorem** states that every planar graph is four-colorable. The proof involved demonstrating 1936 unavoidable reducible configurations. Obviously, we aren't going to work through each configuration. It was the first major theorem utilizing computation as part of its proof, using it to show all configurations were reducible. No smallest counterexamples exist because they must contain, yet do not contain, one of these configurations. This contradiction means there are no counterexamples at all and that the theorem is therefore true.

2 Planarity Testing

There's a straightforward, but not optimal, algorithm for testing whether a graph is planar. We will consider H-**fragments** of G, where H is a subgraph of G. An H-fragment is either an edge not in G whose endpoints are in H or a component of $G - V(H)$ together with the edges and vertices of attachment that connect it to H. We have an algorithm by Demoucron-Malgrange-Pertuiset that uses such H-fragments to incrementally build a planar embedding of G. We consider only biconnected graphs here, since a graph is planar if all of its biconnected subgraphs are planar, so we can simply run this procedure on each block (which we can determine in linear time through a BiCC algorithm like Hopcroft-Tarjan).

```
procedure DMP-PlanarityTest(Biconnected Graph G(V, E))
    G_0 ← arbitrary cycle in G
    i ← 0
    while G_i ≠ G do
        B ← G_i-fragments of G
        for all b ∈ B do
            F(b) ← faces of G_i that contain all vertices of attachment for b
            if F(b) = ∅ then
                return false
        if ∃b ∈ B : |F(b)| = 1 then
            h ← b
        else
            h ← randomly select any b ∈ B
        P ← path between two vertices of attachment of h
        G_{i+1} ← embed P across a face in F(h)
        i ← i + 1
    return true
```

There exist linearly optimal algorithms, such as one using DFS (again) by our good friends Hopcroft and Tarjan. However, such algorithms are a bit more complex and beyond the scope of the course.