1 More on Line Graphs

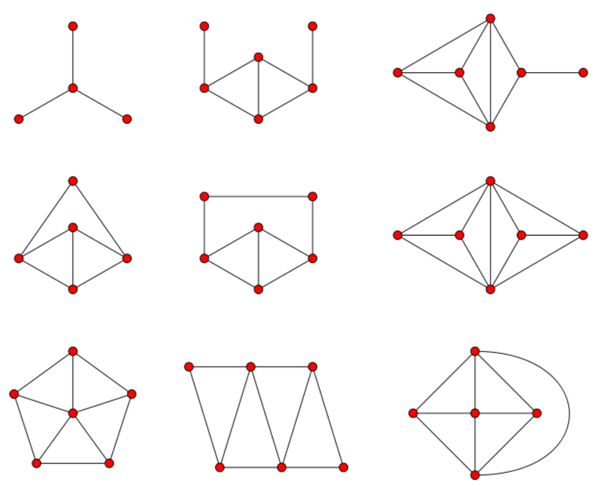
As we discussed in the previous class, there exists a relation between certain problems on line graphs and problems on an original graph. E.g., if you recall, identifying a maximum matching on a graph corresponds to a maximum independent set on a graph.

Let's say that we wish to identify a maximum independent set on a general graph. Computing a maximum independent set is computationally more difficult than a maximum match. So, we can potentially simplify our problem if we're able to identify some graph H such that G is the line graph of H, or $L(H) = G$. If we can do that, then we can solve a maximum match on H and easily translate the solution to G. The question then becomes, does there exist such an H? Below we're characterize some conditions we have of G such that a corresponding H exists.

For a simple graph G, there is a solution to $L(H) = G$ if and only if G decomposes into complete subgraphs, with each vertex of G appearing in at most two of these complete subgraphs.

A **double triangle** is an induced subgraph of graph G that consists of two triangles sharing an edge and no edge existing between the vertices that comprise the third vertex of each triangle. A triangle T is **odd** if $\exists v \in V(G) : |N(v) \cap V(T)|$ is odd. For a simple graph G, there is a solution to $L(H) = G$ if and only if G is claw-free and no double triangle of G has two odd triangles.

The graphs below list all **forbidden subgraphs**. For a simple graph G, there is a solution to $L(H) = G$ if and only if G does not contain any forbidden subgraph as an induced subgraph.



Being able to identify the graph H in $L(H) = G$ can be done in linear time. However, a discussion of such an algorithm is beyond the scope of the course.

2 Hamiltonian Cycles

We've discussed spanning cycles several times previously. Now, we're going to go a bit more in-depth.

Not all graphs have a spanning cycle. Graphs with a spanning cycle are called **Hamiltonian Graphs**. Such a cycle is called a **Hamiltonian Cycle**. What properties must a Hamiltonian graph have? We're going to consider simple and connected (obviously) graphs here.

- A Hamiltonian graph must be biconnected.
- A Hamiltonian bipartite graph must have equal sizes sets.
- If $c(G)$ is the number of components of a graph G, then Hamiltonian graph G must satisfy $c(G-S) \leq |S|$ for all possible $S \subseteq V(G)$.

These give us several necessary conditions. But what about sufficient conditions; i.e., what properties must a graph G have in order to be Hamiltonian? We'll discuss these in the next (and final!) class.