# 1 More Hamiltonian Cycles

In the previous class, we talked about a few necessary conditions that Hamiltonian graphs, graphs containing a spanning cycle, satisfy. We're now going to look at a few more conditions. For these conditions, consider all discussed graphs as simple and connected.

If $G$ has at least three vertices and $\delta(G) \geq \frac{n}{2}$, then $G$ is Hamiltonian.

If $\forall u, v \in V(G), (u, v) \notin E(G), d(v) + d(u) \geq n$, $G$ is Hamiltonian if and only if $G + (u, v)$ is Hamiltonian.

A **closure** of a graph $G$, $C(G)$, is the graph with vertex set $V(G)$ obtained from $G$ by iteratively adding edges joining pairs of nonadjacent vertices whose degrees sum to at least $n$, until no such pair remains. A graph $G$ is Hamiltonian if and only if its closure $C(G)$ is Hamiltonian. The closure of $G$ is also well-defined.

Consider graph $G$ with vertex degrees $d_1 \leq \cdots \leq d_n$, where $n \geq 3$. If $i \leq \frac{n}{2}$ implies that $d_i > i$ or $d_{n-1} \geq n - i$, then $G$ is Hamiltonian. This would further imply that the closure of $G$ is $K_n$. So if we were to ever compute the closure of a graph to be a clique, then we'd know that the original graph is Hamiltonian.

A **Hamiltonian path** is a spanning path. A join between two graphs $G$ and $H$, $G \vee H$, is the graph created by adding edges between all vertices of $G$ with all vertices of $H$. Or if $J = G \vee H$, $V(J) = V(G) \cup V(H)$ and $E(J) = E(G) \cup E(V) \cup \{\forall u \in V(G), \forall v \in V(H) : (u, v)\}$. A graph $G$ has a Hamiltonian path if and only if $G \vee K_1$ has a Hamiltonian cycle.

For all graphs except $K_2$, if the connectivity of the graph is larger than or equal to the independence number, $\kappa(G) \geq \alpha(G)$, then $G$ has a Hamiltonian cycle.