1 4-Color Theorem

Last class it was proven that planar graphs can be properly vertex-colored with at most 5 colors. Let's try and reduce our bound to four colors. An approach for the proof involves us seeking a minimal counterexample, which is an unavoidable set of graphs that can't be present in a four-colored planar graph. We consider triangulations, since every simple planar graph can be contained in a triangulation.

A **configuration** in a planar triangulation is a separating cycle C with the portions of G inside of C. A configuration is **unavoidable** if a minimal counterexample must contain a member of it. A configuration is **reducible** if a planar graph containing it cannot be a minimal counterexample.

Since the minimal degree of a planar graph is less than or equal to five and in a triangulation the minimum degree is greater than or equal to three, then we immediately have three unavoidable configurations. For these, C is length three, four and five each with single vertex in the middle attached to each vertex in C.

We can try and prove that planar graphs are four-colorable using these configurations. However, the configuration using a cycle of length five isn't shown as reducible, so larger configurations must be considered.

The **Four Color Theorem** states that every planar graph is four-colorable. The proof involved demonstrating 633 unavoidable reducible configurations. Obviously, we aren't going to work through each configuration. It was the first major theorem utilizing computation as part of its proof, using it to show all configurations were reducible. No smallest counterexamples exist because they must contain, yet do not contain, one of these configurations. This contradiction means there are no counterexamples at all and that the theorem is therefore true.

2 Line Graphs

The **line graph** of G, written as $L(G)$, is the simple graph whose vertices are the edges of G. Edges $v, u \in E(G)$, represented as vertices $v, u \in V(L(G))$, have an edge between them in $L(G)$ if they share a common endpoint $w \in V(G) : \{(v, w), (u, w)\} \in E(G)$. There is a relationship between problems involving edges in G and problems involving the vertices in $L(G)$:

1. An Eulerian Circuit in G is a spanning cycle in $L(G)$.
2. A matching in G is an independent set in $L(G)$.
3. A cut edge $e = (u, v)$ in G is a cut vertex in $L(G)$ if $d(u), d(v) > 1$.

4. Edge-coloring in G is equivalent to vertex coloring in $L(G)$.

Exploiting these relationships can be extremely useful. Note for example, it's possible to compute a maximum matching in polynomial time while a maximum independent set requires exponential time. Likewise, as we'll discuss later, the same might be said of finding an optimal vertex coloring in general takes exponential time while an optimal edge coloring only requires polynomial time.

We going to focus on the last problem today. Edge-coloring in G and how it relates to vertex coloring in $L(G)$.

3 Edge-coloring

Edge coloring is the problem of assigning labels, i.e. *colors*, to all $e \in E(G)$ such that no two edges have the same color if they share an endpoint. We use similar terminology as with vertex coloring. A coloring is **proper** if it satisfies the above criteria. We consider a k-**edge-coloring** to be a proper edge coloring of k colors. The **edge-chromatic-number** or **chromatic index**, $\chi'(G) = k$, is equal to the smallest k for which G is properly k-edge-colorable. Let's consider some bounds on $\chi'(G)$.

Since all edges incident on the largest degree vertex require separate colors, obviously $\chi'(G) \geq \Delta(G)$.

If we consider a greedy scheme to color edges and note that no edge shares endpoints with more than $2\Delta(G) - 1$ edges, we have the bound $\chi'(G) \leq 2\Delta(G) - 1$.

As a greedy edge coloring scheme on G is equivalent to a greedy vertex coloring scheme on $L(G)$, we further have the bounds $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$.

If G is bipartite, we can show that $\chi'(G) = \Delta(G)$.

For any simple graph, we can further show that $\chi'(G) \leq \Delta(G) + 1$. Or combined with our lower bound, $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.