Lecture 3 - Walks and Induction

Tuesday, February 2, 2021 12:22 PM

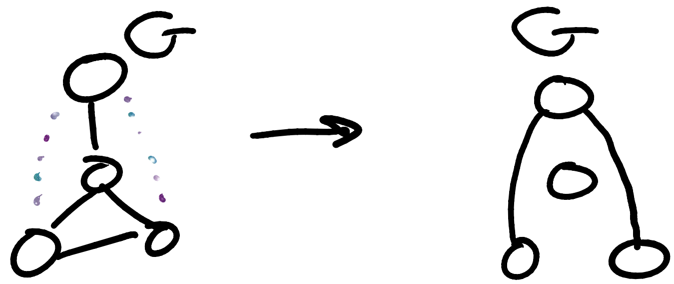
More definitions:

<u>complement</u> of $G \to \bar{G}$ (G simple)

$V(\bar{G}) = V(G)$

s.t. = such that

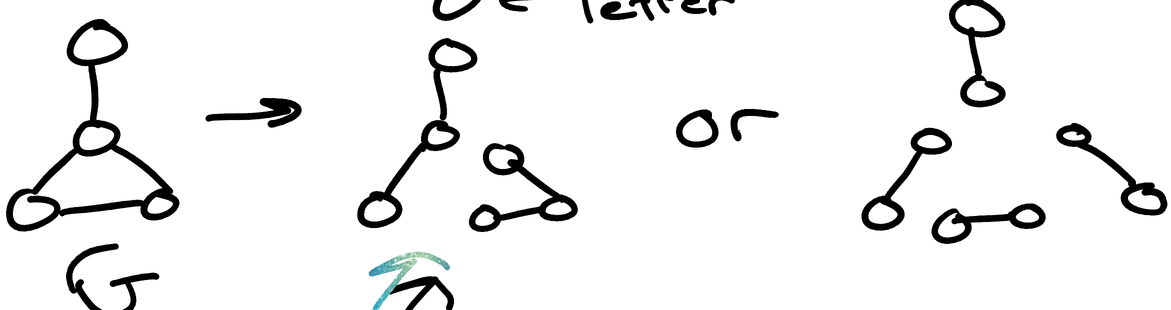
$E(\bar{G}) = \forall u,v \in V(G)$ s.t. $(u,v) \notin E(G)$



Decomposition of G

→ a set of subgraphs s.t. each edge of G appears exactly once within this set

D ← letter



$D = \{(\text{edges}), (\text{edges})\}$

$D = \{P_3\ P_3 \dots P_3\}$

P_3 = path w/3 verts

Time for a walk

Walk: a list of vertices and/or edges, such that each listing is adjacent to the listing proceeding and preceeding it

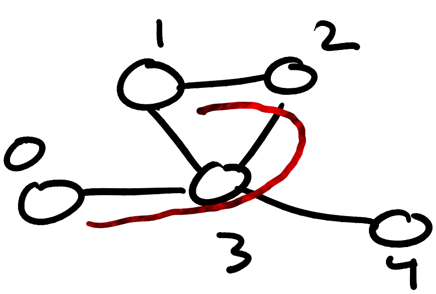
$$W: \{v_0, e_1, v_2, e_5, v_7\}$$

$$W: \{e_1, e_5\}$$

→ Note: we can repeat vertices and edges

Trail: a walk, but we don't repeat edges

Path: a trail, but we don't repeat vertices



$$W: \{0, 3, 2, 1, 3, 2, 1, 3, 4, 3, 0\}$$

$$T: \{0, 3, 2, 1, 3, 4\}$$

$$P: \{0, 3, 2, 1\}$$

Length: number of edges in W / T / P

Length: number of edges in our W / T / P

Hop: traversing a single edge

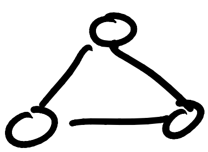
u,v-path: a path that starts at u and ends at v

u,v-walk/trail: same idea

<u>closed</u> path/walk/trail: start and end vertices are the same

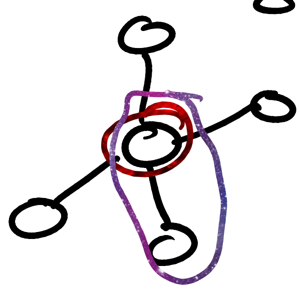
closed path = a cycle

Triangle

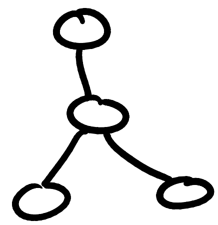


C_3, K_3

Star graph $K_{1,n}$



$K_{1,4}$



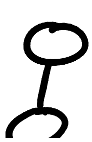
claw $\cong K_{1,3}$

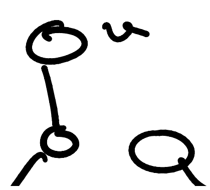
Biclique

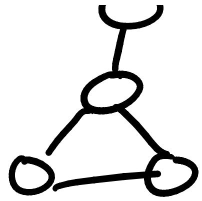
Let's get connected

G is <u>connected</u> if $\forall u,v \in V(G)$

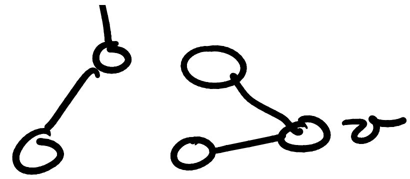
$\exists$ u,v-path







G is connected

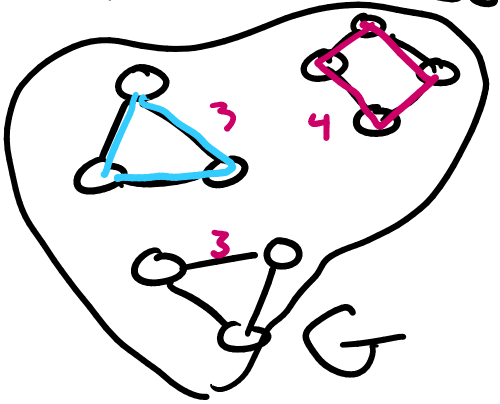


G is disconnected

__connected component__: a __maximal__ connected subgraph of G

maximal: can't be made bigger

maximum: biggest of all possibilities



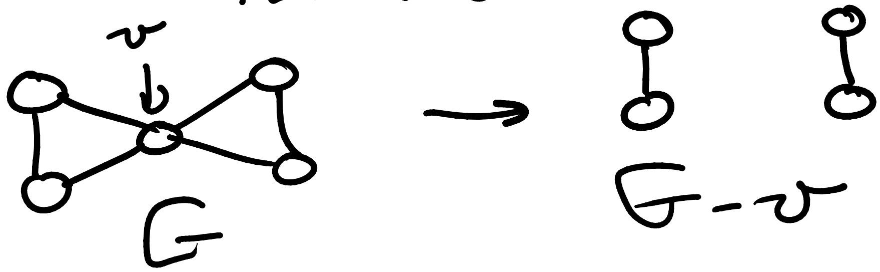
Same idea:

minimal

and

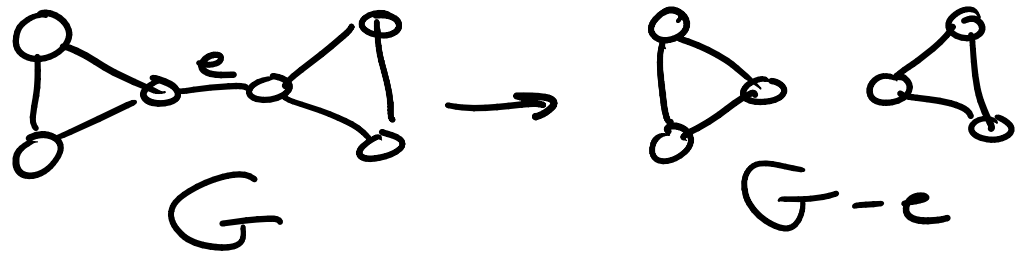
minimum

cut-vertex: some $v \in V(G)$ such that $G - v$ has more conn. components than G



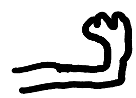
cut-edge: some $e \in E(G)$ s.t. $G - e$ has more C.C.s

$G-e$ has more C.C.s than G



Time for the meat


Chicken leg

Weak induction 

Prove $2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$

Basis: $P(n=1) \Rightarrow 2^{1+1} - 2 = 2 = 2^n \checkmark$

Inductive step: $P(n = k+1)$

- Assume $P(k)$ is true (Inductive hypothesis)

$\rightarrow$ show $P(k+1)$ is true

$$P(k+1) = \underbrace{2^1 + 2^2 + \ldots + 2^k}_{} + 2^{k+1}$$

$$2^1 + 2^2 + \ldots + 2^k = 2^{k+1} - 2$$

if this is true $\rightarrow$ Plug into $k+1$ case

$$P(k+1) = 2^{k+1} - 2 + 2^{k+1}$$

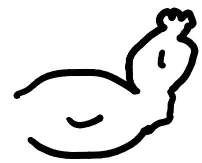
$$= 2^{k+2} - 2 = 2^{n+1} - 2 \quad \square$$

Weak induction:

$P(1), P(2), P(3) \ldots P(k), P(k+1)$

show this — assume this — prove this

Strong induction:



$P(1), \ldots, P(k), \ldots, P(n)$

show this — assume this for all $1 \le k < n$ — prove this using $P(k)$ and induction hypothesis

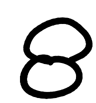
Example proof:

Every closed odd walk contains an odd cycle

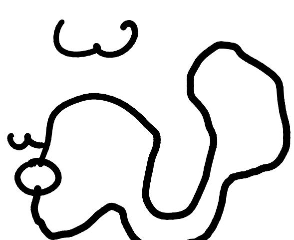
odd => odd # edges

Induction on ℓ = length of our walk

Basis: $P(\ell = 1)$:



Inductive step: $P(n > k \geq 1)$



Assume we have a walk of length n, walk is closed

Consider multiple cases:

Case 1: walk has no repeated vertices

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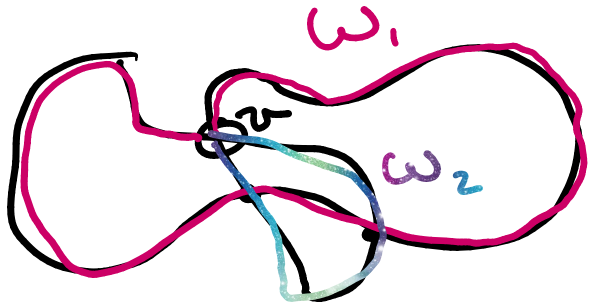
=> just an odd cycle

Case 2: some vertex $v \in W$ repeats



→ implies we can break up $W = W_1 + W_2$

- Consider the length of W_1 $\quad W_1 = \text{odd}$, $W_2 = \text{even}$

note: $|W_1| < n = |W|$

Power of induction (inductive hypothesis)

=> $\exists$ odd cycle $\in W_1$

=> this odd cycle is also in W $\square$

Necessity and sufficiency

also: equivalence relation

A if and only if B

A iff B

$\boxed{A \Leftrightarrow B}$ to prove → Show $A \Rightarrow B$, $B \Rightarrow A$

to prove $\boxed{B \Rightarrow A}$

$G \in$ some class

P = some property

Necessity: if $G \Rightarrow P$

Sufficiency: if $P \Rightarrow G$

Prove (Friday):

G is bipartite iff G contains no odd cycles