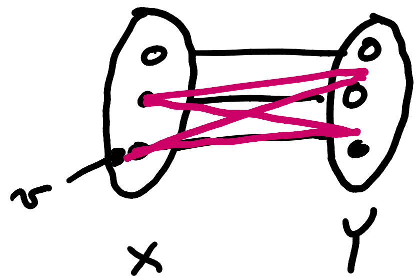
G is bipartite $\Longleftrightarrow$ G has no odd cycles

$\underline{G \text{ is bipartite}} \Longrightarrow G$ has no odd cycles



- consider all possible paths from v

→ Note: odd paths end on some vertex in Y and even end in X

$\Longrightarrow$ any closed path is even

$\square$

G $\underline{\text{has no odd cycles}} \Longrightarrow G$ is bipartite

- Assume G is connected
- consider some $v \in V(G)$

define two sets X, Y

define $f(u)$ = shortest u,v-path length

$X = \{\text{all } a \in V(G) : f(a) = \text{even}\}$

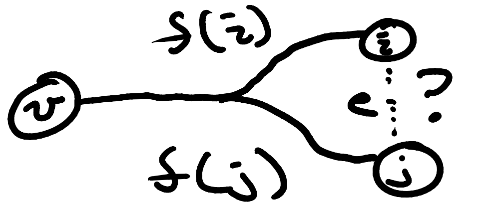
$Y = \{\text{all } b \in V(G) : f(b) = \text{odd}\}$

? Q: are X, Y <u>independent</u>

? Q: are X, Y indep

(no edge between vertices in the set)

Consider shortest paths from v to two vertices in X or two in Y



note: $f(i), f(j)$ both even or both odd

even + even = even
odd + odd = even

→ adding edge e creates an odd walk

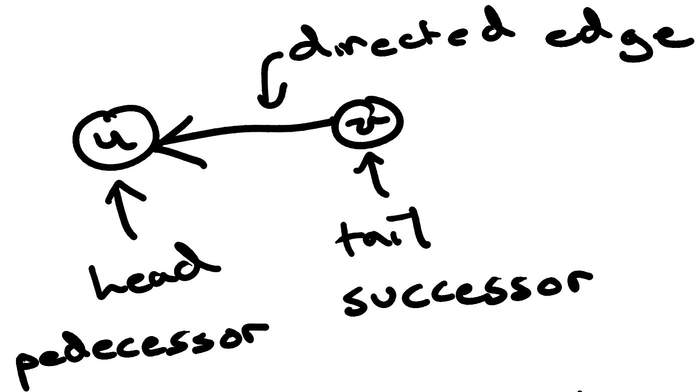
→ every odd walk has an odd cycle

⇒ existence of e would be a contradiction

⇒ both X, Y are independent sets □

Directed Graphs

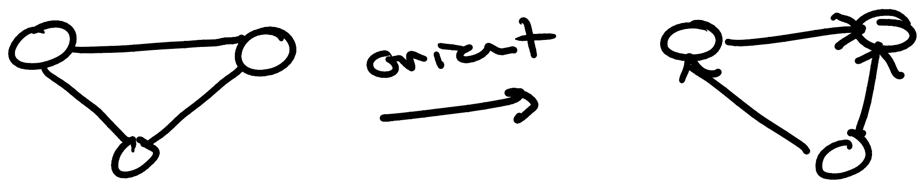
- All edges with directionality



... directionality

pedecessor

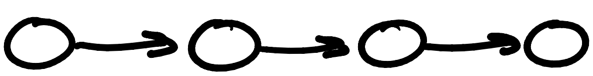
Orientations: adding directionality to <u>all</u> edges in an undirected graph



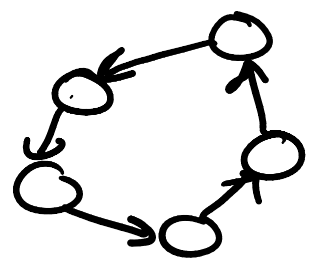
G D

walks, paths, trails, cycles

=> same definition as with undirected graphs, but we now follow the edge direction

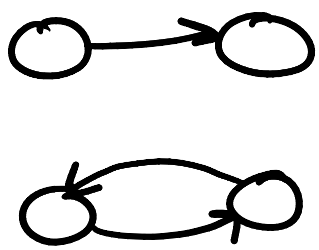


directed path

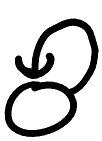


directed cycle

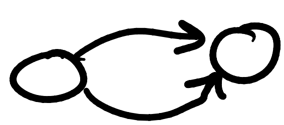
simple graph



simple digraphs



self loop



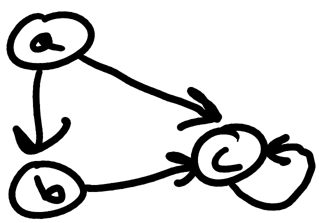
<u>not</u> simple

self loop

simple

Adjacency matrix: no longer symmetric

a nonzero at a_{ij} implies an edge from $i \to j$



$$\begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\Sigma \, row_i =$ out degree of i

$\Sigma \, colown_i =$ in degree of i

Out degree $d^+(v) = \#$ of edges where v is the tail

in degree $d^-(v) = \#$ of edges where v is the head

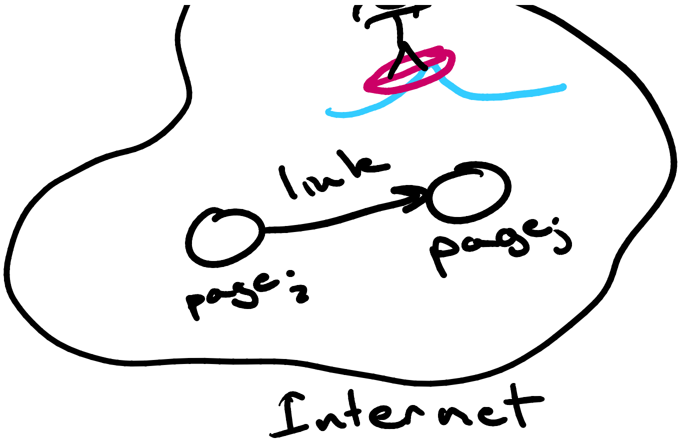
$N^+(v) =$ out neighborhood

$N^-(v) =$ in neighborhood

PageRank

Agent-based: random surfer model

→ random surfer is randomly clicking through



is randomly clicking through links

→ random walk through the web graph

→ PageRank: probability distribution of where the surfer is currently (which web page)

Issue: sources : $d^-(v) = 0$

sinks : $d^+(v) = 0$

Solution: consider randomly jumping to any other vertex in the graph

---

Graph Algorithmic model

Standard graph algorithm:

- Vertices and/or edges have some saved "state"
- For some # of iteration:
  update v's state based on the states of v's neighborhood

on the states of v's neighborhood

(See notes for algorithm)

Linear Agebraic Model

For sinks: we would explicitly have to add those edges to all $v \in N(v)$

consider our adjacency matrix A

consider our diagonal degree matrix D

- D has nonzeros along diagonal
- D_{ii} = degree of vertex i
- $D_{ij} = 0$, $i \neq j$

define transition probability matrix

M = defines how pagerank flows to vertices in the matrix

$$M = (D^{-1}A)^T$$

Note: inverse of a diagonal matrix is just inverse of each entry along the diagonal each entry along the diagonal

consider initial pageranks as a vector $p_0 = \begin{bmatrix} 1/n \\ \vdots \\ 1/n \end{bmatrix}$ $n = |V(G)|$

We calculate updates as

$$p_{i+1} = M p_i$$

We have a steady state solution

$$p_\infty = M p_\infty$$

Recall for eigenvector v of some matrix A we have

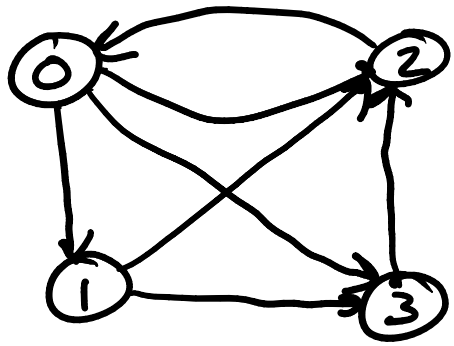
$$Av = \lambda v$$

$\Rightarrow$ pageranks are simply the eigenvector for the M matrix with eigenvalue $\lambda = 1$

Example calculation

0 1 2 3

Example calculation



$$A = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \end{array}$$

sum of row

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = (D^{-1}A)^T \quad (D^{-1}A) = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

sum of each row $= 1$ ✓

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

$$p_0 = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

← PR of vertex 0

← PR of vertex 3

$$P_1 = M p_0$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/ \end{bmatrix}$$

vertex 5

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 6/24 \\ 2/24 \\ 11/24 \\ 5/24 \end{bmatrix}$$