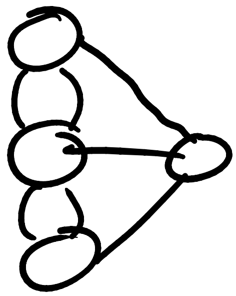
Lecture 5 - Euler Tours and Havel-Hakimi

Tuesday, February 9, 2021 12:20 PM

Recall Bridges of Königsberg



→ ? → Does a closed trail exist containing all edges (Euler tour)

First: Prove if $\forall v \in V(G) : d(v) \geq 2$

$\Rightarrow \exists C_n \in G$ for some n

To do this: we'll use the power of <u>Extemal arguments</u>

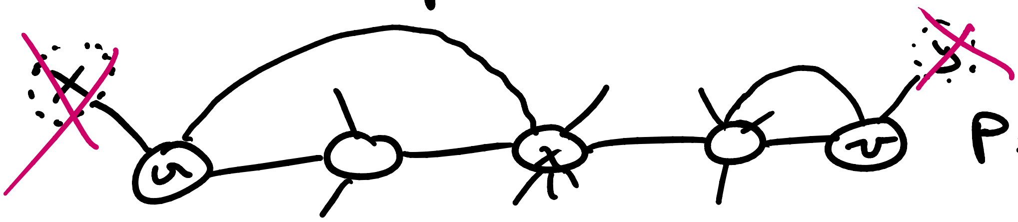
a.k.a. the Extremal Principle

$\exists$ within a set some subset that is maximum/maximal and minimum/minimal for some countable well-ordered property

$\forall v \in V(G) : d(v) \geq 2 \Rightarrow \exists C_n \in G$

... Some maximum length

Consider: Some maximum length path $P \in G$



$P = u,v$-path

Note degrees of v and u are g.t.e. 2

Can there exist some $x \in V(G)$ s.t. $x \notin P$?

$\to$ NO, by choice of P

otherwise x,v-path is longer than $P \Rightarrow$ contradiction :) wooo

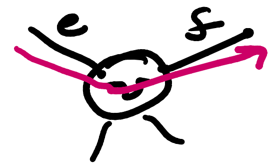
This implies u's neighbor x must be in $P \Rightarrow u,x$ lie on a cycle

$\square$

G is Eulerian $\Leftrightarrow$ G has at most 1 nontrivial component and $\forall v \in V(G): d(v) = \text{even}$

If G is Eulerian $\Rightarrow$ G has at most 1 nontrivial comp. and $\forall v \in V(G): d(v) = \text{even}$

G has a tour, so for every vertex we visit along edge e, we must leave along some edge f



$\Rightarrow$ all $d(v) = \text{even}$

Trivial to see G must have at most 1 nontrivial component

$\forall v \in V(G): d(v) = \text{even}$ $\Rightarrow$ G is Eulerian
1 nontrivial comp

Induction on $m = |E(G)|$

Basis: $P(m=0) \Rightarrow 0 \Rightarrow \{\ \}$
trivial tour

$P(m > k)$: consider graph G

G has $\forall v \in V(G): d(v) = \text{even}$
and at most 1 n.t. comp.

From our last proof:

$\exists\, C_n \in G$

Inductive step:

Consider $H = G - C$

$|E(H)| = k - p(h)$ case

Note: H must have all even degrees

→ every vertex on the cycle has exactly 2 subtracted from their degree

BUT: H may be disconnected with multiple nontrivial comps.

HOWEVER: we can still invoke I.H. on all components of H

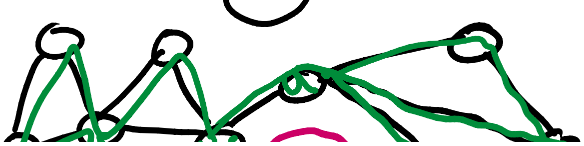
Proof by algorithm to bring it back

Trivial case: $G \cong C$, H is null

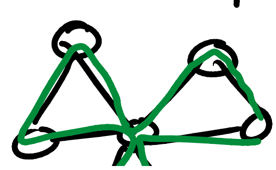
$\Rightarrow$ Euler tour on G is just C

Nontrivial case: H has some number of components → I.H. on all components

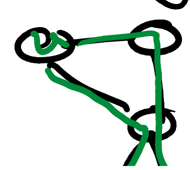
G

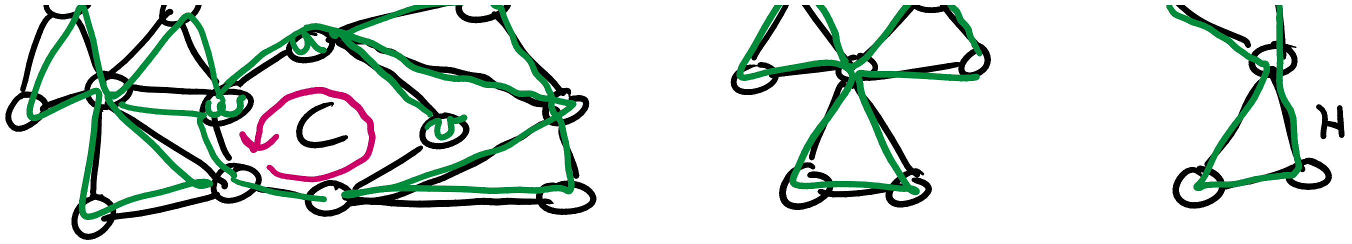


H



G





From I.H. $\Rightarrow$ all components of H have a Euler Tour

To complete the proof, algorithmically combine these tours with C to give us a tour on G

Our algorithm:

start on some $v \in C$

if $d(v) = 2$, continue along C

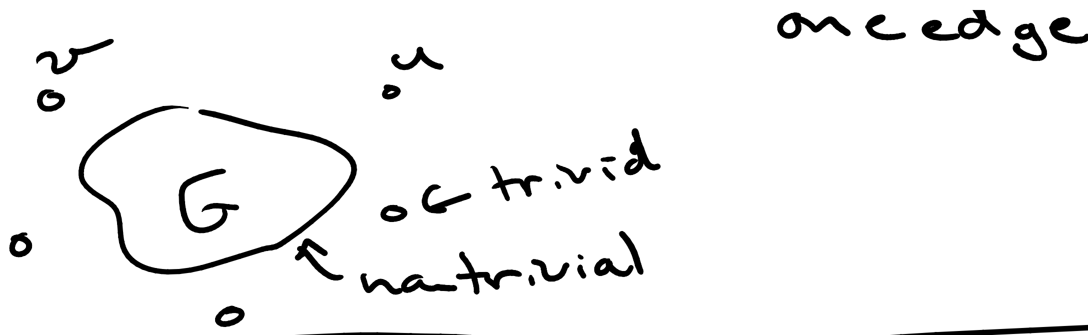
else $\exists$ a Euler tour from v on a component of H (via I.H.)

continue along C

$\rightarrow$ Output is a Euler tour on graph G

Trivial component $\rightarrow$ single vertex

nontrivial $\rightarrow$ multiple verts w/ at least one edge



Degrees

recall $n = |V(G)|$, $m = |E(G)|$

degree of $v \in V(G) \rightarrow d(v)$

d_v

For graph G:

maximum degree $\rightarrow \Delta(G)$

minimum degree $\rightarrow \delta(G)$

G is k-regular if

$$\Delta(G) = k = \delta(G)$$

Note: all cycles are 2-regular

all cliques K_n are $(n-1)$-regular

Degree sum formula:

$$\sum_{v \in V(G)} d(v) = 2m = \text{even}$$

$$\sum_{v \in V(G)} d(v) = 2m = \text{[illegible]}$$

Why: each edge adds +1 to the degrees of each vertex its incident on

for directed graphs:

degrees → $d^+(v)$, $d^-(v)$
out degree, in degree

max/minimum

$\Delta^+(G)$, $\Delta^-(G)$ → max out/in degree

$\delta^+(G)$, $\delta^-(G)$ → min out/in degree

Degree sum formula for digraphs:

$$\sum_{v \in V(G)} d^+(v) = m = \sum_{v \in V(G)} d^-(v)$$

Why? Every outie must have an innie for each edge

+1 $d^+(u)$ +1 for $d^-(v)$

$$u \longrightarrow v$$