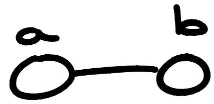
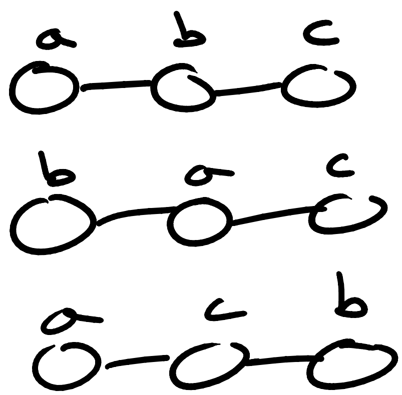
Lecture 7 - Tree Enumeration and Graceful Graphs

Tuesday, February 16, 2021 12:19 PM

Cayley's formula: there exists n^{n-2} possible trees for $|V(T)| = n$



$2^{2-2} = 1$



$3^{3-2} = 3$

Prüfer code: a sequence of labels for tree T s.t. the length of the sequence is $n-2$ and T's vertices comprise the sequence

$$a = \{a_1\ a_2\ a_3 \dots a_{n-2}\}$$

$a_i \in S \qquad f(T) = a$

S = vertex labels of T

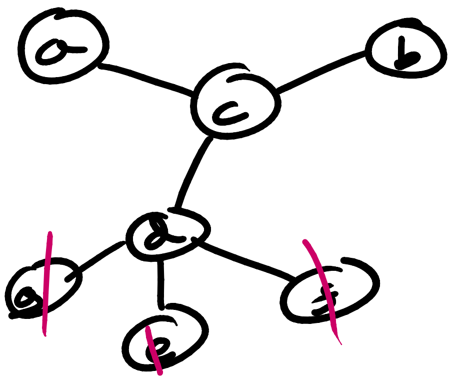
sortable

CreatePrüfer(T) ← sortable
consider in increasing

$a = \emptyset$

for $i = 1 \ldots (n-2)$

- ℓ = label of least remaining leaf in T
- $T = T - \ell$
- a_i = remaining neighbor of ℓ



$$\begin{array}{l} \quad\;\; g \;\; f \;\; e \;\; d \;\; c \;\; b \;\; a \\ S = \{a, b, c, d, e, f, g\} \end{array}$$

$$a = \{\underset{a_1}{c}, \underset{a_2}{c}, d, d, \underset{a_{n-2}}{d}\}$$

↑ prüfer code of (T, S)

CreateTree(a, S)

$V(T) = S$

$E(T) = \emptyset$

My

Consider all S as "unmarked"

for $i = 1 \ldots (n-2)$

x = least unmarked in S that is also not in $a_i \ldots a_{n-2}$

mark x in S

$E(T) \leftarrow (x, a_i)$

$E(T) \leftarrow (x, a_i)$

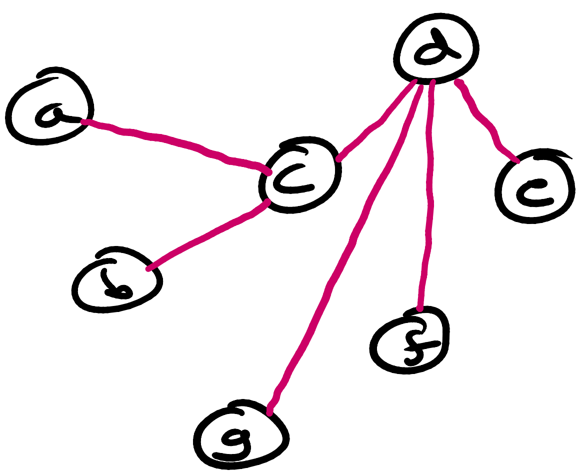
$x, y \leftarrow$ remaining unmarked vertices in S

$E(T) \leftarrow (x, y)$

$a = \{c\ c\ d\ d\ d\}$

$a_1\ a_2\ a_3\ a_4\ a_5$

$S = \{\cancel{a}\ \cancel{b}\ \cancel{c}\ d\ \cancel{e}\ \cancel{f}\ g\}$



Takeaway → for a given tree T and vertex set S, we define a unique code a

AND given code a and vertex set S, we can construct a unique tree T

$f(T) = a$

$f(T) = a$

bijective one-to-one mapping

Bring it on back to Cayley:

$\longrightarrow \exists\ n^{n-2}$ possible trees

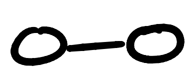
Why? There are n^{n-2} ways to write a prüfer code $a = \{a_1 \dots a_{n-2}\}$

Proofer code Prüf (Cayley's Formula)

For sequence S of vertex labels where $|S| = n \Rightarrow \exists\ n^{n-2}$ possible trees where $V(T) = S$

Note: we're going to prove the uniqueness and existence of the Prüfer code $f(T) = a$ mapping

We'll prove this using strong induction on $n = |S|$

Basis $P(n=2)$  $2^0 = 1$

$f(T) = a = \{\}$

Consider $P(n>2) \Rightarrow T$

Consider tree T with $V(T) = S$, $|S| = n$

Consider x as least element of S where x is a leaf in T

Consider a_1 as neighbor of x

Now $P(k<n)$

$T' = T - x$

$S' = S - x$ Note: $a_1 \in S'$

$a' = \{a_2 \dots a_{n-2}\}$

By I.H. a' exists and is unique given T', S'

Going from $a' \to a$, $T' \to T$

consider that $S' \to S$ we push x to the front of S' to get S

We add back edge (x, a_1) to go from $T' \to T$

$\Rightarrow$ From our createPrüfer algorithm the vertex x and edge (x, a_1)

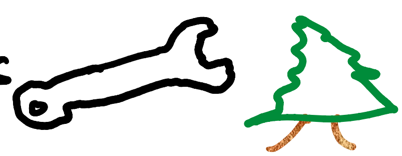
the vertex x and edge (x, a_1) would be first selected for deletion $\to$ first value in a is a_1 NO MATTER WHAT

$\Rightarrow$ The rest of a is simply a' $\quad a = \{a_1, \{a'\}\}$ $\quad f(T) = a$

$\Rightarrow$ So there exists a unique $f(T) = a$ for a given $S = V(T)$

$\Rightarrow \exists\, n^{n-2}$ possible tree configuration for $V(T) = S$
$|S| = n$ $\square$

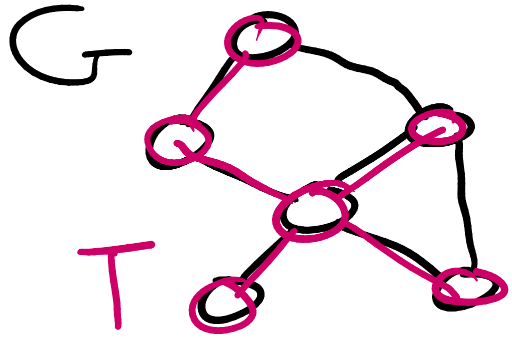
Spanning trees



Spanning tree is a tree subgraph T of some graph G s.t. $\forall v \in V(G)$ $\to \exists\, v \in V(T)$

* spanning trees are acyclic connected subgraphs containing all vertices

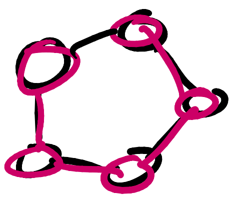
subgraphs containg all vertices of the full graph



Note: the number of spanning trees of a complete graph is n^{n-2}

$\tau(G)$ = # of spanning trees on G

$\tau(K_n) = n^{n-2}$



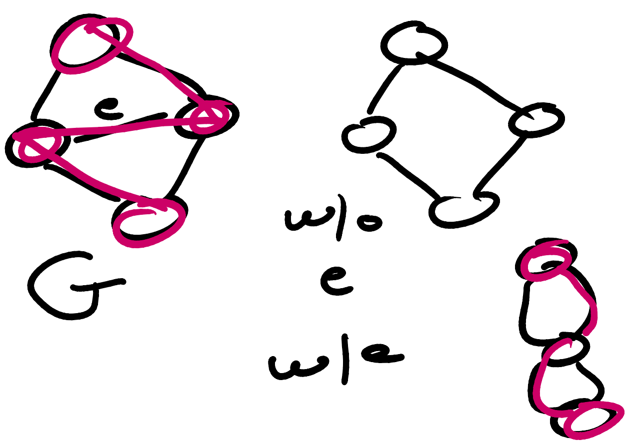
$\tau(P_n) = 1$ $\quad$ $\tau(T) = 1$

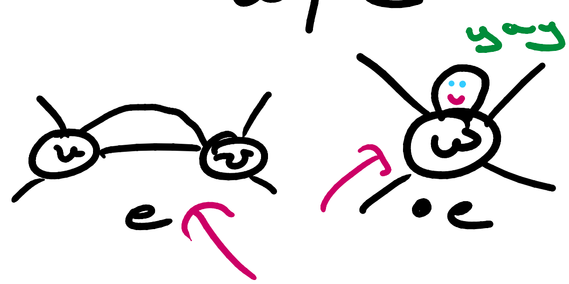
$\tau(C_n) = n$

Let's define a recurrence: (sp?)

$$\tau(G) = \tau(G-e) + \tau(\overset{\text{edge contraction}}{G \cdot e})$$

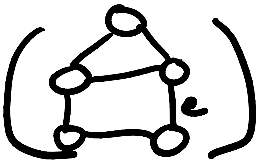
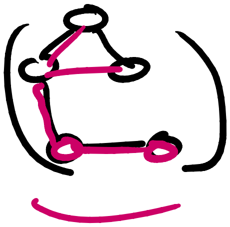
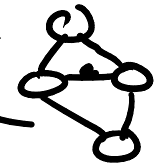
S.T.s $\qquad$ # S.T.s w/o e $\qquad$ # S.T.s w/ e

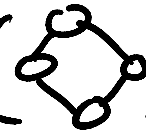




w/ = 

$$\tau(G) = \tau(G - e) + \tau(G \cdot e)$$

$\tau\big($  $\big) = \tau\big($  $\big) + \tau\big($  $\big)$

$= 3 + \tau\big($  $\big) + \tau\big($  $\big)$

$3 + 4 + \tau\big($  $\big) + \tau\big($  $\big)$

$$3 + 4 + 2 + 2 = \boxed{11}$$

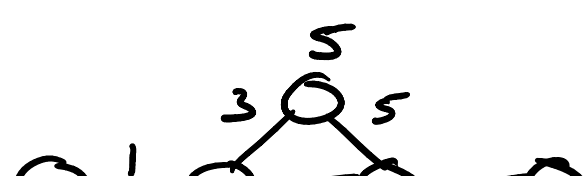
Graceful graph → has graceful labeling

Graceful labeling: a labeling of vertices and edges of G s.t.

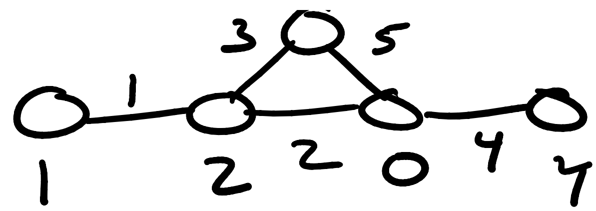
$\forall v \in V(G) : \ell(v) = 0 \ldots m = |E(G)|$ is unique

$\forall e = (u,v) \in E(G) : |\ell(u) - \ell(v)|$ is unique

Graceful:



Graceful



Graceful Tree conjecture:
(Ringel-Kotzig)

All trees are graceful

(unproven)

Caterpillar graphs are graceful
and paths

a graph with all edges incident or on a single path