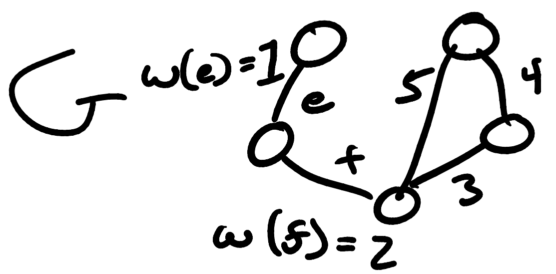
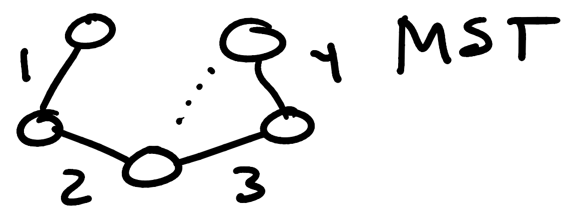
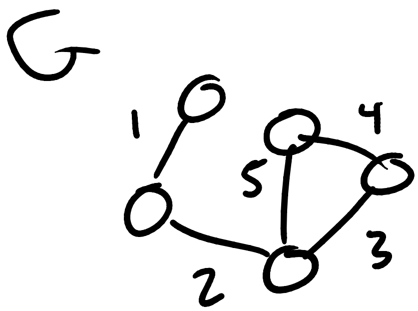
Weighted edges: consider our edge set → each edge has some associated weight

Graph $G = (V, E, w)$

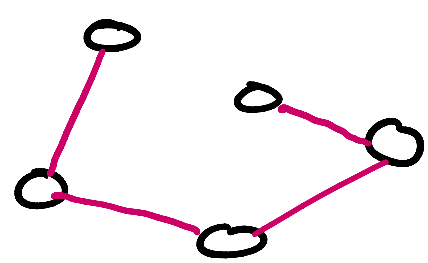


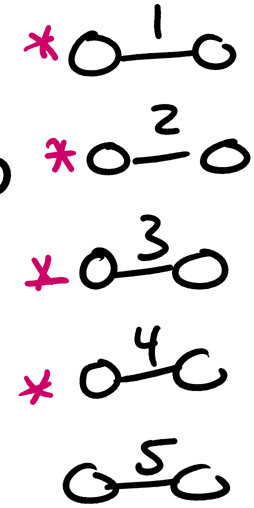


Minimum spanning tree: spanning tree on a edge-weighted graph that has the smallest possible sum of weights



→ T





Prove: Kruskal outputs a M.S.T.

T: consider each edge we add

I: consider each edge we add

→ always joins two separate components

=> cut edge so doesn't introduce a cycle ✓

S: We assume G is connected

on we go until we're done

=> our T spans G ✓

M: pseudo-algorithmic argument

Consider that Kruskal outputs some T s.t. T is not M.S.T. where $T^* =$ M.S.T.

Consider some $e \in E(T), e \notin E(T^*)$ where e is first such edge chosen

Consider adding e to T^* → creates cycle C

Consider $e' \in C$, $e' \notin E(T)$

Note: T^* has all edges in T that were selected before e

→ so e and e' were both available

→ so e and e' were both available for selection by T ⟹ $w(e) \le w(e')$

define $T' = T^* + e - e'$

Note: $W(T') \le W(T^*)$
sum of weights

⟹ we have T' with more edges in common with T than T^*

⟹ repeat the above argument

$T' \to$ converges to T □

Single-source shortest paths SSSP

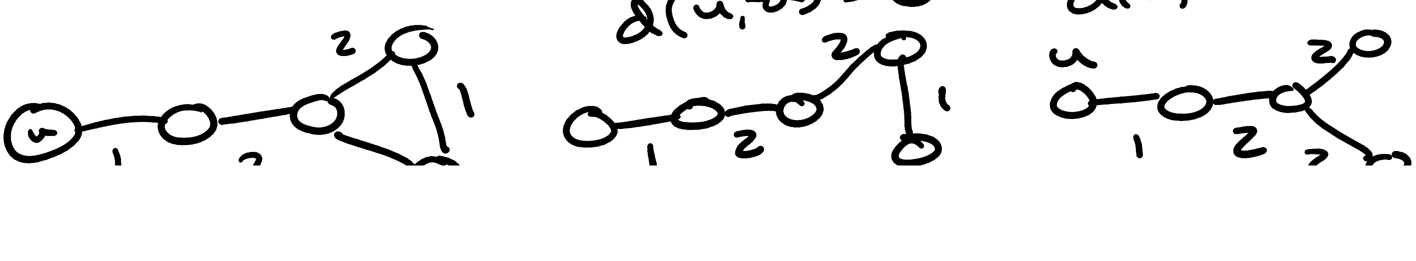
→ from vertex u, indentifying all shortest u,v-paths $\forall v \in V(G)$

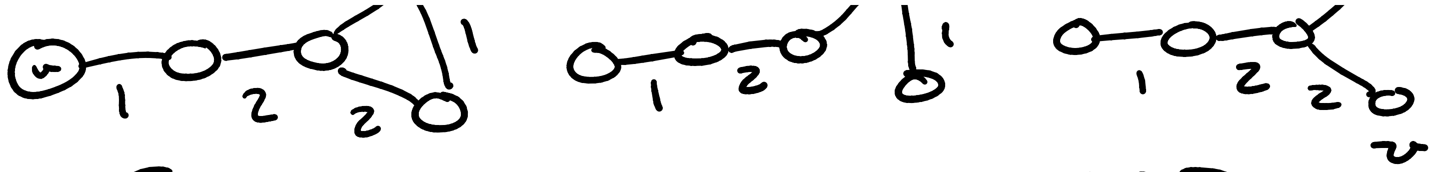
we care about distances $d(u,v)$ as well as the paths themselves

→ shortest paths tree

Note: <u>not</u> equal to M.S.T.

$d(u,v) = 6$ $\quad$ $d(u,v) = 5$

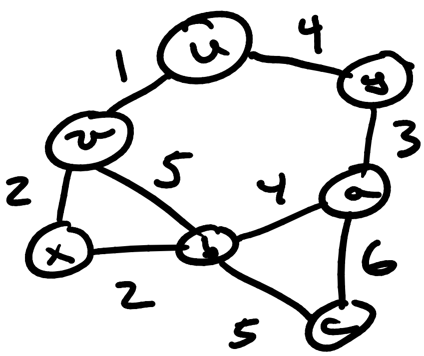




G MST SPT

shortest paths tree

Example of SSSP via Dijkstra



	D_0	D_1	D_2	D_3	D_4	D_5
u	0	0	0	0	0	0
v	∞	1	1	1	1	1
x	∞	∞	3	3	3	3
y	∞	4	4	4		
a	∞	∞	∞	∞		
b	∞	∞	6	5		
c	∞	∞	∞	∞		

exercise 4 reader

Show: correctness/optmality of Dijkstra

Prove: at each iteration

1 - $\forall v \in X \quad D(v) = d(u,v)$ ← actual shortest u,v-path length

2 - $\forall v \notin X \quad D(v)$ is shortest u, v-path length from X

X = set of vertices visited during our algorithm

via weak induction on $|X|$

$P(1) \Rightarrow X = \{u\} \quad D(u) = d(u,u) = 0$

$P(1) \Rightarrow X = \{u\} \quad D(u) = d(u,u) = 0$

all $v \in N(u)$ take on distance of (u,v) edge

$P(k) \Rightarrow |X| = k$

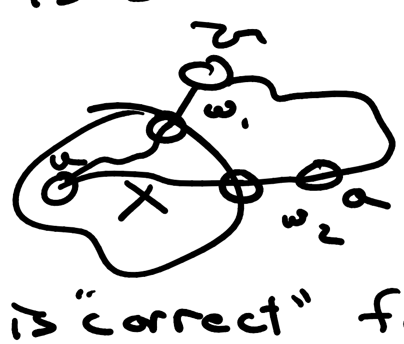
assume via I.H. that the above holds

$P(k+1) \Rightarrow X' = X + v$

v is selected s.t. $D(v)$ is least of all $v \notin X$

First show: $D(v) = d(u,v)$

By I.H. $\rightarrow$ shortest path directly from X to v is $D(v)$, so any other path that exits X and reaches v is bounded below by $D(v)$ ✓



Secondly show: $D(w)$ is "correct" for $X' = X + v$, $\underbrace{w \notin X'}$

Via I.H. $\rightarrow$ $D(w)$ is shortest u,w-path distance directly from X

weight of edge

We update $O(w) = \min\Big(\underbrace{O(w)}_{\text{from } X},\ \underbrace{O(v) + \overset{\text{weight of edge } (w,v)}{W(w,v)}}_{\text{from } X'}\Big)$

→ shortest possible path directly through X', as v is the only way to get to w through a vertex not originally in X

$\square$