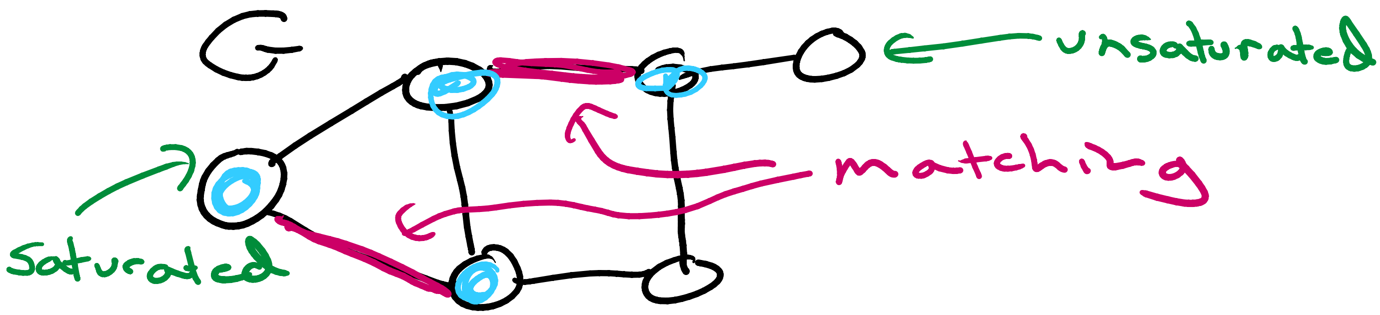
Lecture 9 - Matching

Tuesday, February 23, 2021 12:29 PM

Match on G - a set of non-loop edges with no <u>shared endpoints</u>

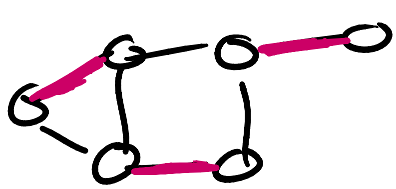


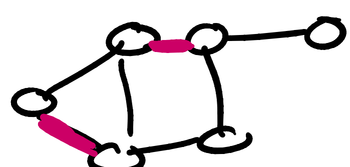
Saturated vertex - endpoint for matched edge

Unsaturated vertex - not endpoint for any matched edge

maximum match - largest possible match on a graph

maximal match - a match that can't be made larger


maximum


maximal

Perfect match - a match that

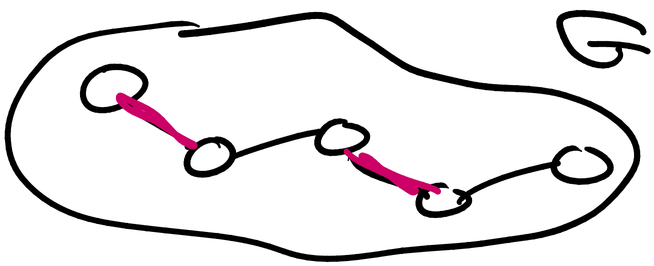
Perfect match - a match that saturates all vertices

$$|M_p| = \frac{|V(G)|}{2} \qquad M = \{e_i \dots e_n\}$$

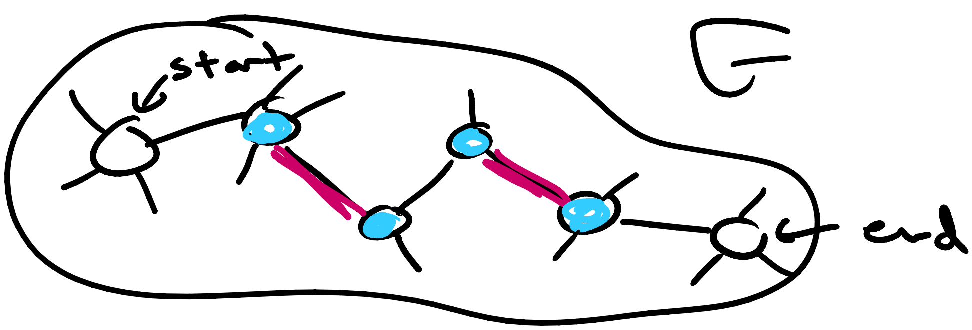
→ only possible if $|V(G)|$ is even

→ only possible if $d(v) > 0 \ \forall v \in V(G)$

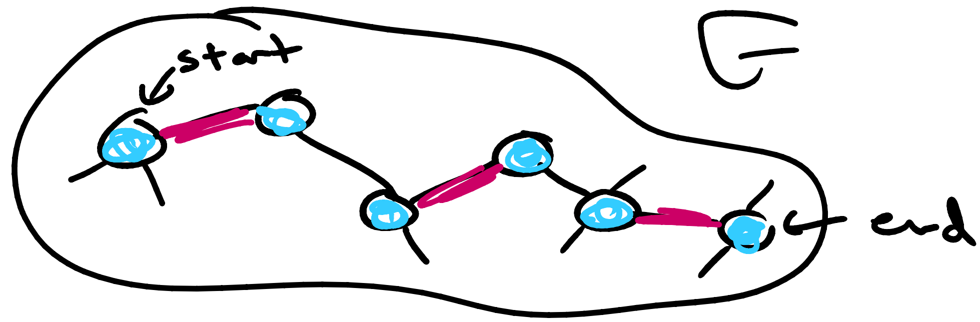
M-alternating path - given a matching M on same G, a M-alt path is a path subgraph were every other edge is in M



M-augmenting path - an M-alt path that has a start and end vertex that are unsaturated



Note: along an M-aug path, we can swap edges in M with those not in M to create a larger match



→ if there exists an M-aug path, there exist some M' where $|M'| > |M|$

Berge: a matching M on G is maximum iff G has no M-aug path

To prove this, we'll be proving the *contrapositive*

$P \to Q \quad \neg Q \to \neg P$ ✓

$P \leftrightarrow Q \quad \neg P \leftrightarrow \neg Q$

If $\exists M'$ on G where $|M'| > |M|$

If $\exists M'$ on G where $|M'| > |M|$

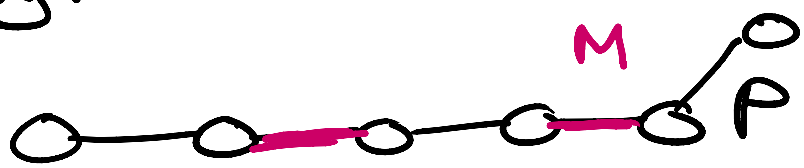
$\iff \exists$ M-aug path on G

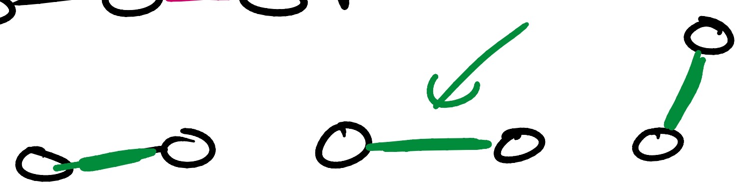
if $\exists$ M-Aug path $\Rightarrow \exists M'$, $|M'| > |M|$

demonstrated this above

- take the <u>symmetric difference</u> of M and the M-aug path to get M' ✓

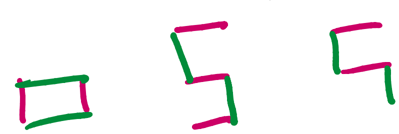
symmetric difference $\Rightarrow$ XOR



$P \triangle M \Rightarrow$ 

If $\exists M'$, $|M'| > |M| \Rightarrow \exists$ M-aug path

- consider $F = M \triangle M'$

Note: F is comprised of cycles and/or paths 

- consider only odd paths in F

as $|M'| > |M|$

as $|M'| > |M|$, $\exists$ at least one

from a parity argument

$\Rightarrow$ this is our M-augmenting path $\boxed{\checkmark}$

Matching on bipartite graphs

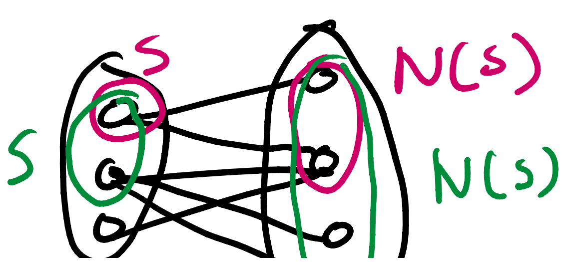
→ we generally consider optimality with respect to the small bipartite set

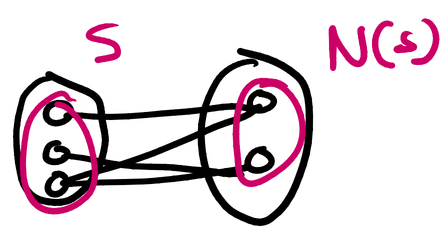
So for $B \Rightarrow V(B) = \{X, Y\}$

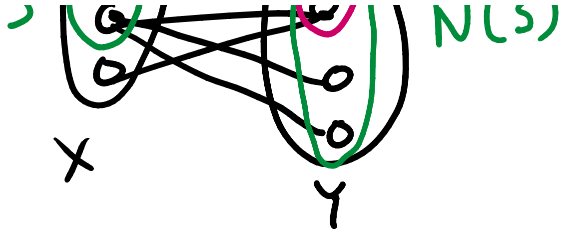
$|X| < |Y|$

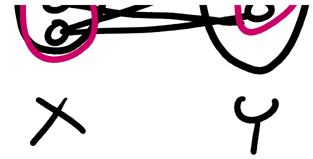
→ $\exists M$ that fully saturates all $v \in X$

Hall: $\exists M$ that fully saturates X where $|X| < |Y|$ on X,Y-bipartite graph iff $\forall S \subseteq X$ $|N(S)| \geq |S|$







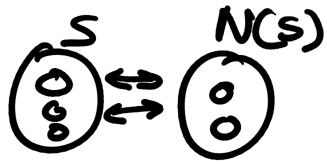


☆ contrapositive ☆

No fully saturating M $\Leftarrow\!\!\!\Rightarrow$ $\exists S \subseteq X$ s.t. $|N(S)| < |S|$

If $\exists S \subseteq X$ s.t. $|N(S)| < |S|$ => no fully X-saturating M

- Demonstrated above



No way to match a larger to a smaller set of vertices ✓

No fully X-saturating M => $\exists S \subseteq X$ s.t. $|N(S)| < |S|$

- consider some maximum M (power of extremal argument)
- consider some $u \in X$, $u \notin M$
- consider S = all $v \in X$ that are

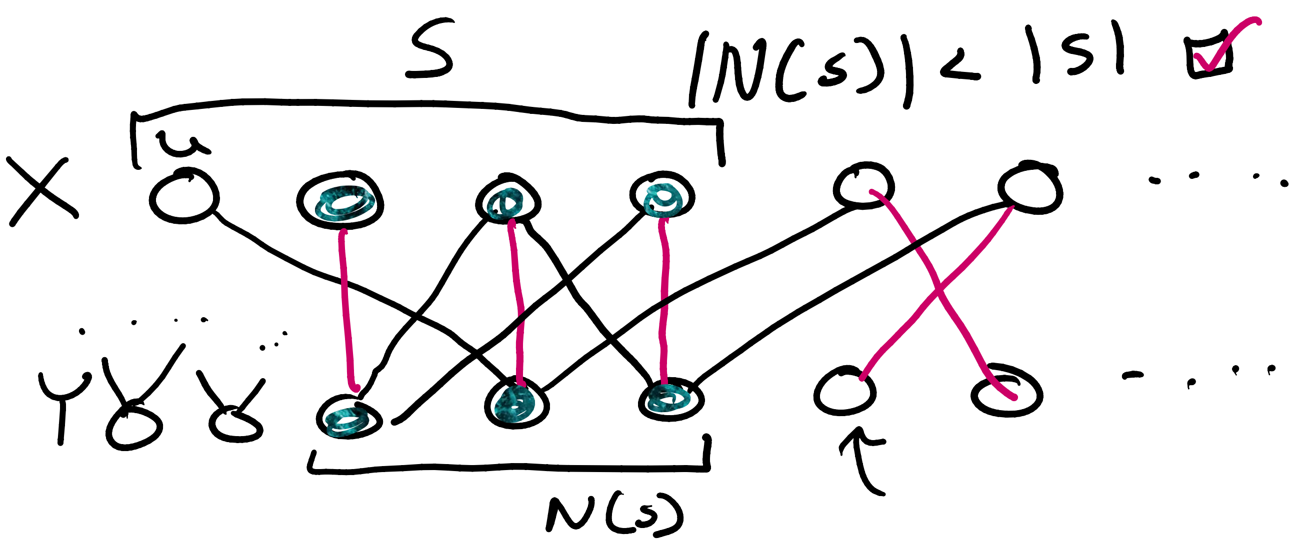
- consider S = all $v \in X$ that are reachable via a u,v-M-alt path

Note: $N(S)$ must be fully saturated

→ if not, we'd have a M-aug path

So considering the bijection from $N(S) \leftrightarrow S - u$, we have

$$|N(S)| = |S| - 1 < |S|$$



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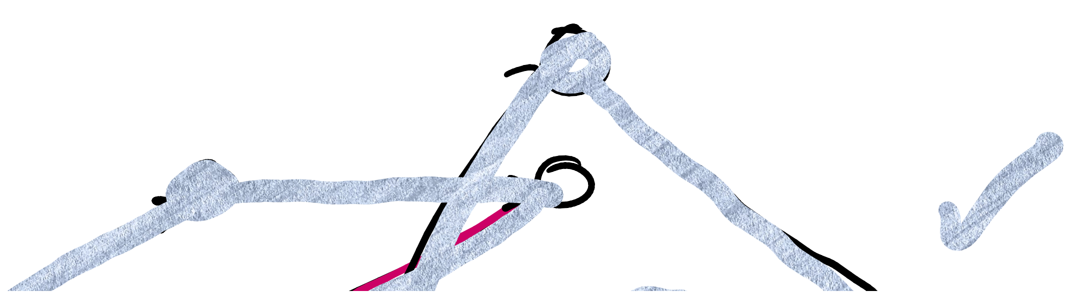
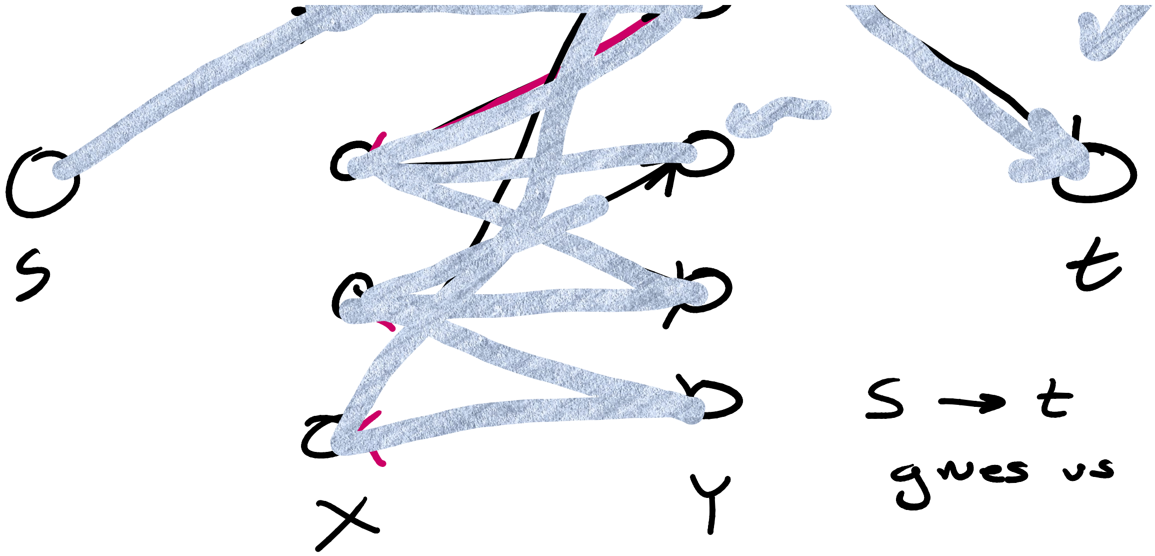
maxBipartiteMatchAlgorithm





s → t
gives us M-aug path