Lecture 10 - General graph matching and covers

Friday, February 26, 2021 2:22 PM

$o(G)$ = number of odd components of a graph G

Tutte's Theorem

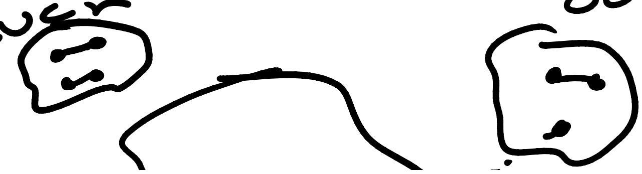
G has a perfect match (P.M.) iff $\forall S \subseteq V(G): o(G-S) \leq |S|$

G has P.M. $\Rightarrow \forall S \subseteq V(G): o(G-S) \leq |S|$

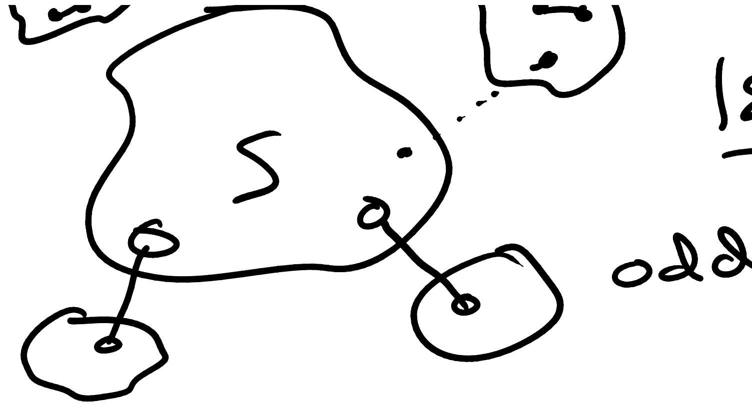
- Consider some arbitrary S
- Consider $G-S$
- Since all odd components of $G-S$ can't have a P.M. contained within them, so at least one vertex in every odd component must be matched with a vertex in S

odd

even



$|S| \geq o(G-S)$



$\underline{|S| \geq o(G-S)}$

$\forall S$

$\forall S \subseteq V(G): o(G-S) \leq |S| \Rightarrow G$ has a P.M

contrapositive

G has no P.M. $\Rightarrow \exists S$ s.t. $|S| < o(G-S)$

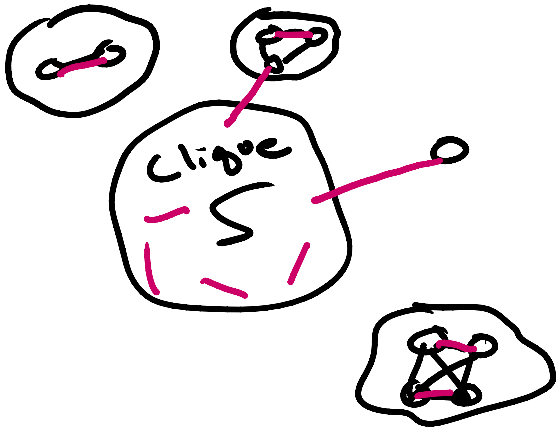
Note: condition holds when adding edges to G

→ we consider an <u>extreme</u> G', where G' is the edge-maximal graph having no P.M.

$\Rightarrow G' + e$ → has a P.M.

- Define $S = \forall v \in V(G')$
$d(v) = |V(G')| - 1$

Case 1: $G' - S \to$ all components are cliques



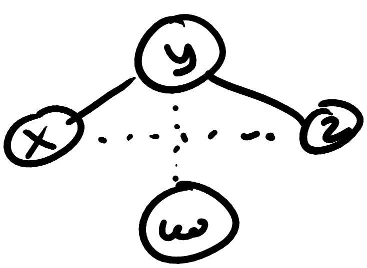
S must be *bad*
$|S| < o(G-S)$

Otherwise, we can construct a P.M. ✓

Case 2: $G' - S$ not comprised of cliques

Note: $\exists x, z$ s.t. $(x,z) \notin E(G-S)$

$\exists y$ s.t. $(x,y), (y,z) \in E(G-S)$



$\exists w$ s.t. $(y,w) \notin E(G-S)$

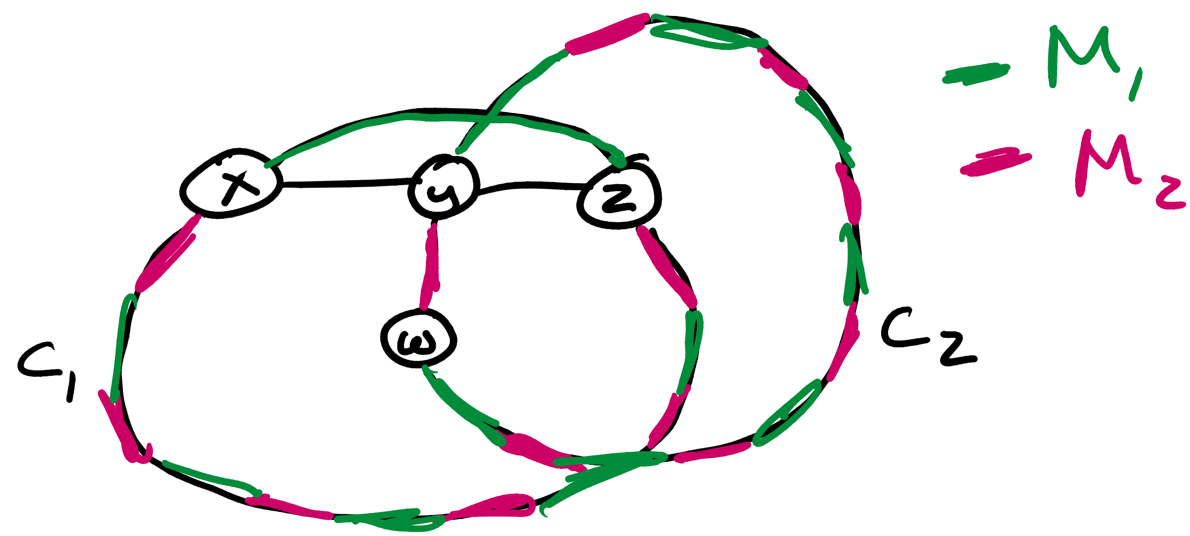
as

Show: adding (x,y) or (y,z) creates a P.M. on G'

→ we can actually show that a P.M. exists on G'

a P.M. exists on $\bar{G}'$

- define:
  - M_1 = P.M. on $G' + (x, z)$
  - M_2 = P.M. on $G' + (y, w)$
  - $F = M_1 \triangle M_2$ must be ~~paths~~ ~~or~~ cycles or single vertices
  - C_1 = cycle with (x, z)
  - C_2 = cycle with (y, w)



Case 2a: $C_1 \neq C_2$

P.M. on G' = all $e \in M_2$, $e \in C_1$
all other $e \in M_1$

$\Rightarrow$ P.M w/o (x,z) or (y,w) ✓

Case 2b: $C_1 = C_2$

P.M. on $G' = M_1$ on C_2 from w until x or z

if we reach x:

- P.M. on $G' + = (x,y) + M_2$ from y to z ✓

if we reach z:

- P.M. on $G' + = (y,z) + M_2$ from y to x ✓

$\Rightarrow$ either way we have a P.M. on G' w/o (x,z) or (y,w)

$\Rightarrow$ Contradiction!

X X X X

So S must be bad

$|S| < o(G-S)$

so S must be bad

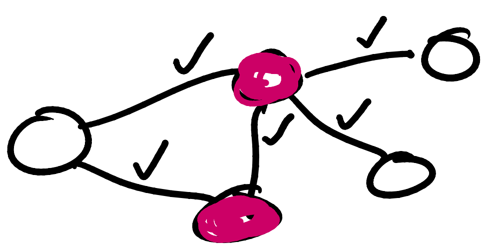
$|S| < o(G-S)$

in all cases ☑

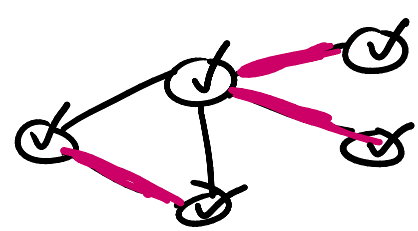
$$\boxed{\text{Tutte: } G \text{ has P.M.} \iff \forall S \subseteq V(G)\ \ o(G-S) \leq |S|}$$

Vertex cover: a set $Q \subseteq V(G)$ that has at least one endpoint in $\forall e \in E(G)$

Edge cover: a set $L \subseteq E(G)$ that has at least one edge incident on $\forall v \in V(G)$



vertex cover



edge cover

König-Egerváry: on bipartite graph $G \implies$ the size of a minimum vertex cover = size of a vertex cover = size of a maximum match

$|M_{max}|$ = max match

$|C_{min}|$ = min cover

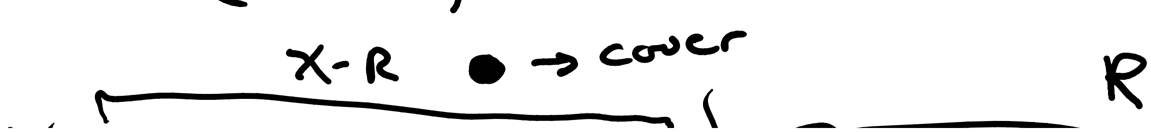
note: $|C| \geq |M|$ for any cover/match

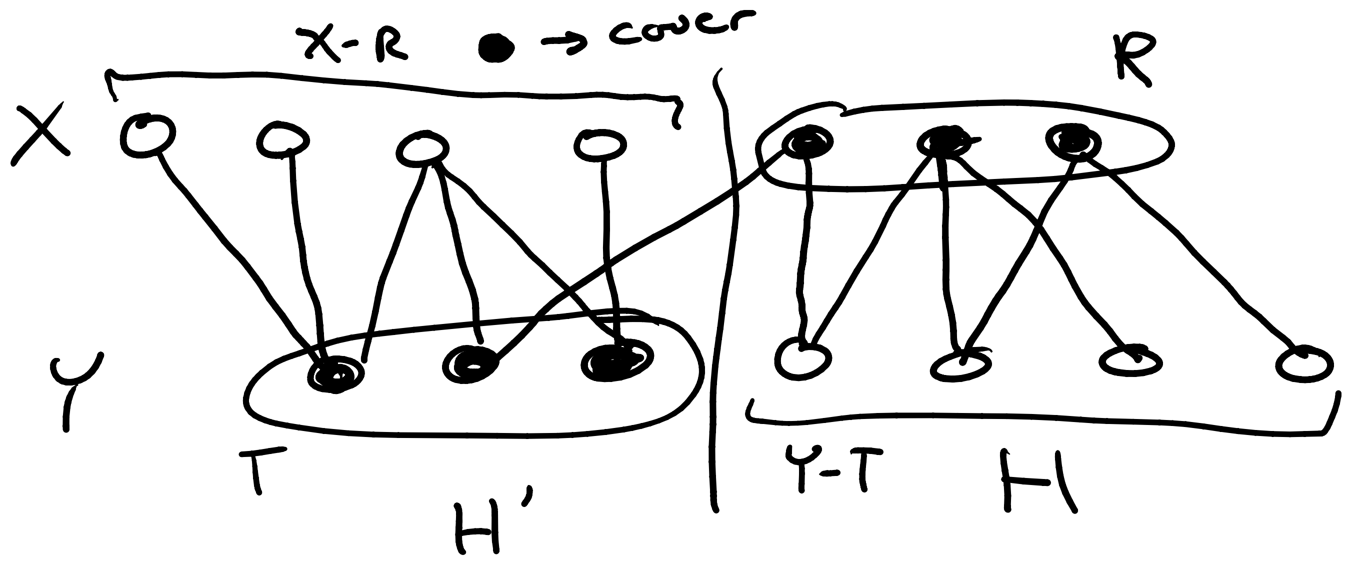
→ every matched edge needs to be covered by one $v \in C$

Show: $|M_{max}| = |C_{min}|$

- define on $G = X, Y$ - bigraph
  - $R = C \cap X$ (C is min cover)
  - $T = C \cap Y$
  - $H = R \cup (Y - T)$ subgraph
  - $H' = T \cup (X - R)$ subgraph

Note: no edge from $(Y-T)$ to $(X-R)$, because $R \cup T$ = Cover





Consider $\forall S \subseteq R$ and $|N_H(S)|$

if $\exists S : |N_H(S)| < |S|$, we could swap $N_H(S) \leftrightarrow S$ in C to get a smaller cover

→ so R satisfies Hall's condition by our <u>extreme</u> choice of C

→ so R is saturated by M

→ likewise, T is saturated by M via same argument

$\Rightarrow$ we have $|M| = |C|$ ✓

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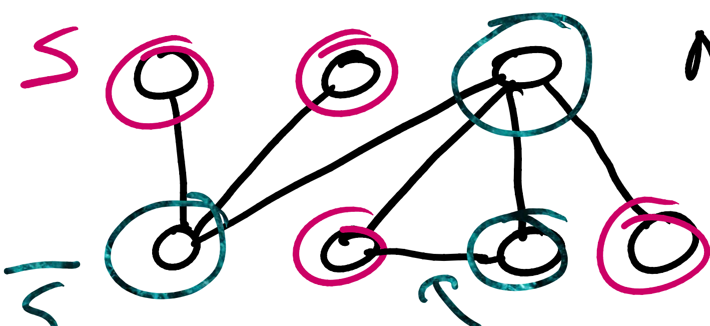
as $|C| \geq |M|$, this is optimal ☑

Independent set - a set of vertices that have no edges between them

Independence number - the maximum size of an independent set



Note: max independent set ≠ size of larger bipartite set in bigraph

Also note:

* S is indep. set iff $\bar{S}$ is a cover

→ a max independent set is the coplement of a min vertex cover

$$|S| + |\overline{S}| = |V(G)|$$