Lecture 11 - Connectivity Introduction

Tuesday, March 2, 2021 2:25 PM

Our bag of tricks

- weak induction $P(1) \ldots P(k) \to P(k+1)$
- strong induction $P(1) \ldots P(k) \ldots P(n)$

  * considering the cases
- external arguments → ∃ max path

  ∃ cycle etc.
- equivalences (iff) → $P \Leftrightarrow Q$

  $P \Rightarrow Q,\ Q \Rightarrow P$

  * Slota easy one way harder the other

  SEOWHTO
- contrapositives → $P \Rightarrow Q$

  $\neg Q \Rightarrow \neg P$

  $P \Leftrightarrow Q,\ \neg P \Leftrightarrow \neg Q$
- structural arguments

  define $S \subseteq V(G)$

  consider $x \in S$ … etc.
- countability / enumerative arguments

  parity even/odd $|V(G)|$

  … $|E(G)|$ $\delta(G)$ … etc

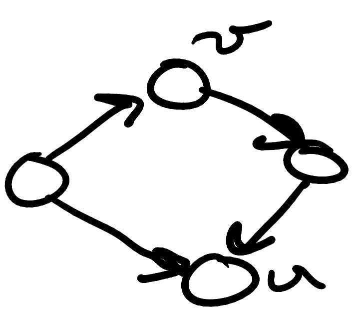
parity even/odd |V(G)|
bounds on $|V(G)|$, $\delta(G)$, ... etc.

Directed connectivity

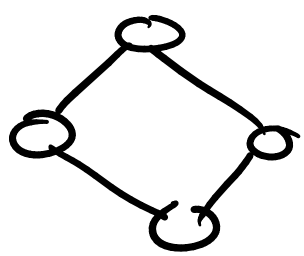
weak connectivity
strong connectivity

weak => a digraph is *weakly* connected if $\exists\, u,v$-path $\forall u,v \in |V(G)|$ when considering the <u>underlying</u> graph

→ ignore directivity of edges

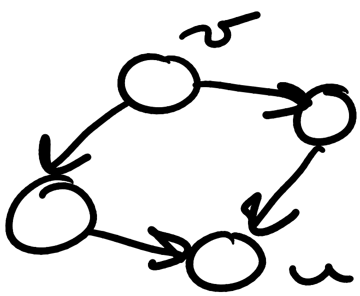


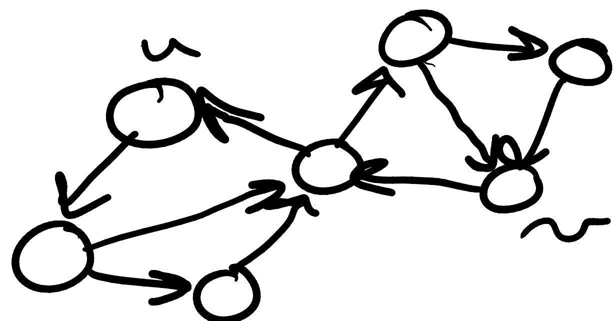
G



underlying graph of G

is weakly connected b/c the underlying graph is connected strong connectivity - a digraph is strongly connected if $\forall u, v \in V(G)$ $\exists$ u,v-path following directions of edges



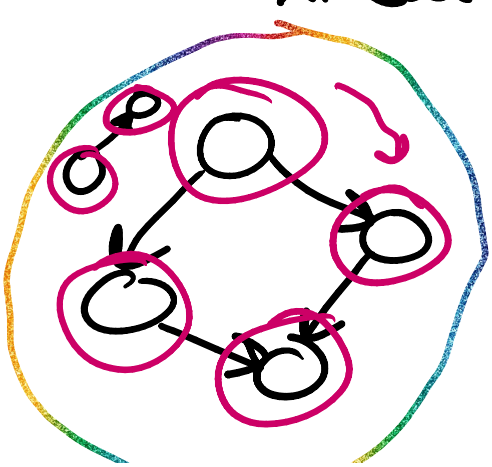


✗ NOT ✗ strongly connected

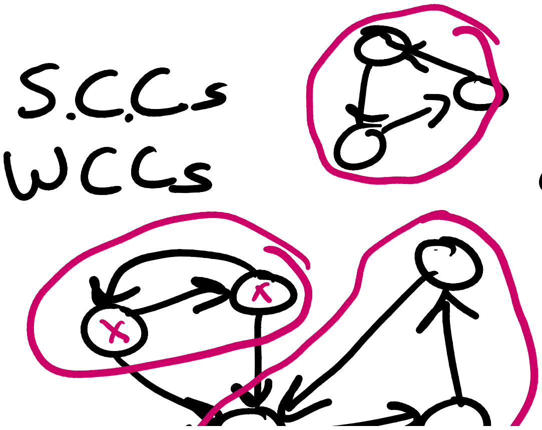
✓ strongly connected ✓

weakly connected component - maximal weakly connected subgraph

strongly connected component - maximal strongly connected subgraph



6 S.C.Cs
2 WCCs



2 WCCs
3 SCCs

Vertex Connectivity

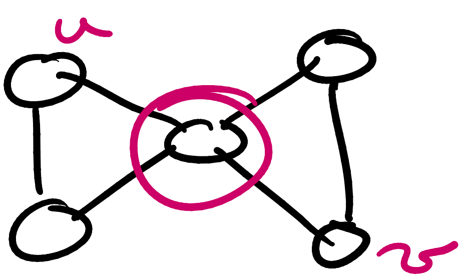
Separating set – a subset $S \subseteq V(G)$ on a connected graph G s.t. $G-S$ is disconnected

or a single vertex

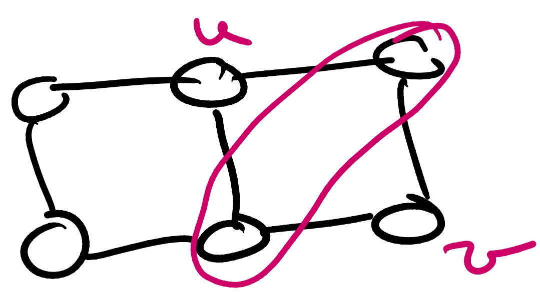
AKA vertex cuts
vertex separators

Connectivity of $G = \kappa(G) = k$ is the size of a minimum vertex separator

$\rightarrow$ G is k-connected



$\kappa(G) = 1$



$\kappa(G) = 2$

$K(G) = 1$ $\qquad$ $K(G) = 2$

G is 2-connected ✓

→ note: clique K_n is $(n-1)$-connected

Edge Connectivity

Disconnecting set - a set of edges $F \subseteq E(G)$ s.t. $G - F$ is disconnected

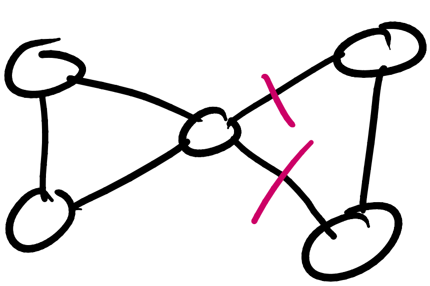
→ usually called edge cut

Bond: minimal disconnecting set

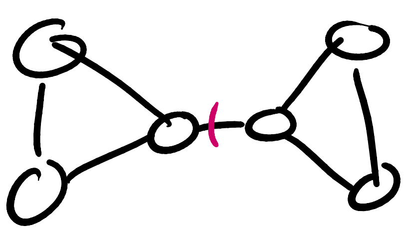
Edge-connectivity of $G = K'(G) \geq k$

- minimum size of a disconnecting set → G is k-edge-connected

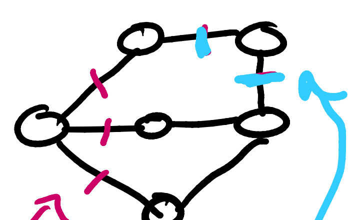
→ Note: not necessarily minimum



$K'(G) = 2$



$K'(G) = 1$



Bond

$K'(G) = 2$ $\quad$ $K'(G) = 1$ $\quad$ 'Bond'

$K(G) = 1$ $\quad$ $K(G) = 1$ $\quad$ min cut

If G is 2-connected, G is aslo 1-connected

Note for later

$$K(G) \overset{?}{\lessgtr} K'(G) \overset{?}{\lessgtr} \delta(G)$$

can we place bounds between these values?

First note: $K'(G) \leq \delta(G)$

Why? Trivially, removing all incident edges on min degree vertex v will disconnect v from G

disconnect G

Now consider $K(G)$ and $K'(G)$

Note: $K(G) \leq |V(G)| - 1$ by definition

Extremal argument

- consider minimum edge cut F that separates G into $S, \bar{S}$

$\bar{S} = G - S$

Case 1: $\forall u \in S, \forall v \in \bar{S} : \exists (u,v) \in E(G)$

$\rightarrow$ we know

$$|K'(G)| = |F| = |S||\bar{S}| \geq |V(G)| - 1$$
$$\geq K(G)$$

Case 2: $\exists x \in S, y \in \bar{S} : (x,y) \notin E(G)$

define $T =$ all $u \in N(x) : u \in \bar{S}$

and all $v \in S - x :$

$\exists (v,z) \in E(G), z \in \bar{S}$

Note: all x,y-paths go through T

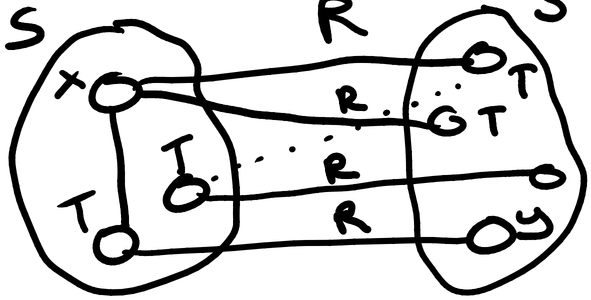
$\rightarrow T$ is a vertex

$\to T$ is a vertex

define $R =$ all $e = (x, w) : w \in T \cap \bar{S}$

and $f = (a, b) : a \in T \cap S, b \in \bar{S}$

(one of each possible)



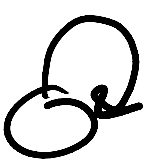
Note: $|R| = |T|$

$|F| \geq |R|$

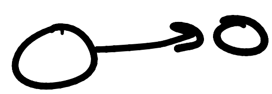
$$K'(G) = |F| \geq |R| = |T| \geq K(G)$$

$$K'(G) \geq K(G) \checkmark$$

$$\underbrace{K(G) \leq K'(G) \leq \delta(G)}_{BB} \quad \square$$



loopy graph



no self-loops

$\Rightarrow$ loopless