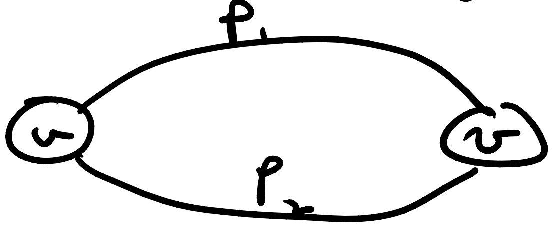
2-connectivity

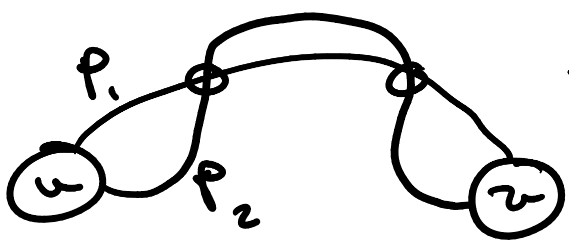
→ must remove at least 2 vertices to disconnect G

internally-disjoint paths



→ u,v-paths that share no internal vertices

internally-edge-disjoint paths



→ u,v-paths that share no internal edges

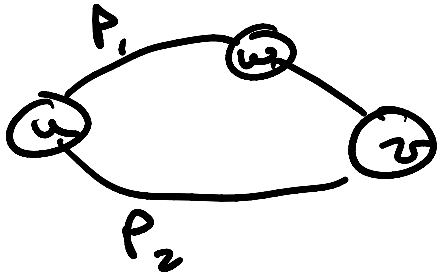
Whitney: G $\overset{|V(G)| \geq 3}{\text{is}}$ at least 2-connected iff $\forall u,v \in V(G): \exists\, u,v$-idps
(internally-disjoints paths)

$\exists P_1 P_2$ idps $\forall u,v \in V(G) \Rightarrow G$ is 2-connected

- considering any $u,v \in V(G)$

- considering any $u,v \in V(G)$
- consider $\forall w \in P_1$ or P_2

$\Rightarrow$ removing w will NOT



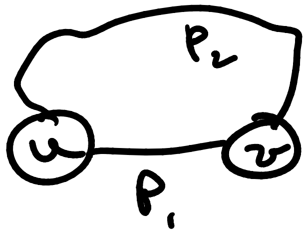
disconnect u from v

$\Rightarrow$ must remove at least 2 vertizes to disconnect u,v ✓

G is 2-connected $\Rightarrow \exists P_1 P_2$ idps

Induction on distance $d(u,v)$

Basis: $d(u,v) = 1$



$P_1 = (u,v)$

$P_2 =$ any other path

as $\kappa'(G) \geq \kappa(G)$, so removing P_1 will not disconnect u from v ✓

- Consider G with u,v s.t.

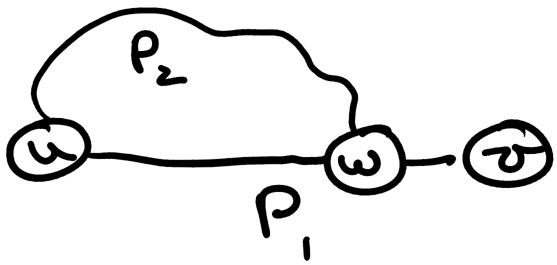
$\underbrace{d(u,v) = n}$,

Assume G is two connecting

Assume G is ...

$\rightarrow \exists$ at least one u,v-path P_1

- consider $w \in P_1$ that is adjacent to v

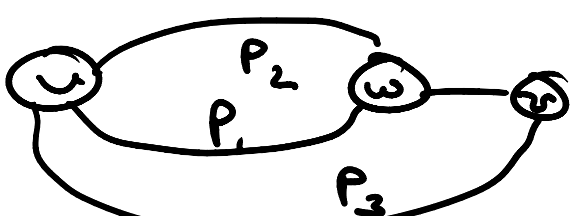


$\rightarrow d(u,w) = k < n$

I.H. $\rightarrow$ 2 internally-disjoint-u,w-paths

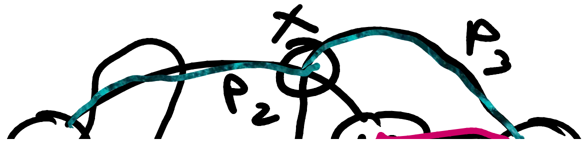
However, as G is 2-connected $\exists P_3 = u,v$-path on $G - w$

Case 1: P_3 does not intersect P_1 or P_2

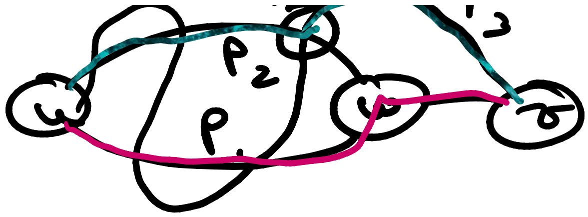


$\rightarrow$ our 2 idps are P_3 and P_1 or $P_2 + (w,v)$ ✓

Case 2: P_3 does intersect P_1 and/or P_2



- consider vertex x the last intersection



– consider vertex x, the last intersecting vertex on P_3 and P_1/P_2

→ wlog lets say $x \in P_2$

→ P_1 and P_2 from u to x $+ (u,v)$ $+ P_3$ from x to v

$\Rightarrow$ 2 idps $\square$

G is 2-connected $\Leftrightarrow \exists P_1 P_2$ idps $\forall u,v \in V(G)$

✓ $\Leftrightarrow G$ is connected and has no cut vertex

✓ $\Leftrightarrow \forall u,v \in V(G) : \exists C$ s.t. $u,v \in C$

$\Leftrightarrow \forall e,f \in E(G) : \exists C$ s.t. $e,f \in C$

Subdivision: $\overset{u}{\circ} \underset{e}{—} \overset{v}{\circ} \rightarrow \overset{u}{\circ} \underset{e_1}{—} \overset{v_e}{\circ} \underset{e_2}{—} \overset{v}{\circ}$

Note: subdividing does not impact 2-connectivity impact 2-connectivity

→ $\exists$ 2 idps $\forall x,y \in V(G)$

Now consider subdividing our e, f to create v_e and v_f

→ we're still 2-connected

→ $\exists\, C \subseteq G$ s.t. $v_e, v_f \in C$

→ $e, f \in C$ ✓  Kewl

---

Ear decomposition

open ear decomposition

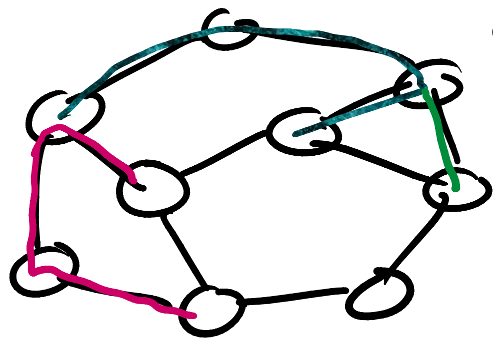
Decomposition on a 2-connected graph G s.t.

$O = P_0\, P_1 \ldots P_k$

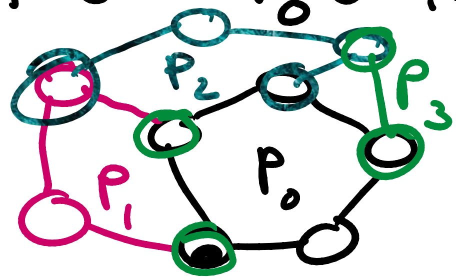
P_0 = cycle

P_i = open path whose endpoints exist on $P_0 \cup P_1 \ldots P_{i-1}$

exist on $P_0 \cup P_1 \ldots P_{i-1}$



G

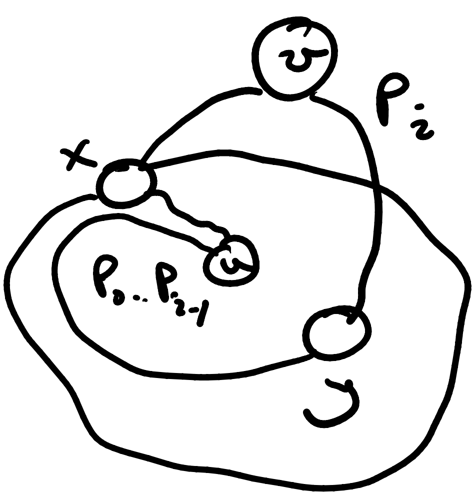


D

G is 2-connected $\iff$ G has an open ear decomposition

G has $D \Rightarrow G$ is 2-connected

- first consider P_0, a cycle
- next consider some P_i
  - consider some $v \in P_i$
  - consider some $u \in P_0 \ldots P_{i-1}$



$\rightarrow$ let's find our 2 idps from v to u

from $v \rightarrow x$, along one x,u-idp

$v \rightarrow y$, along one y,u-idp

y,u-idp

$\Rightarrow$ adding an ear does not affect 2-connectivity ✓

2-connected $\Rightarrow \exists$ D

- let's build a decomposition
- consider $\forall u, v \in V(G)$ and all C, s.t. $u, v \in C$
- Select some C as our P_0

While $\exists e \in E(G)$, $e \notin P_0 \dots P_{i-1}$

- consider any $f \in P_0 \dots P_{i-1}$
- $\exists C$, s.t. $e, f \in C$
- Create P_i by following C in both directions from e until we get to vertices $x, y \in P_0 \dots P_{i-1}$

$\Rightarrow$ this builds our open ✓

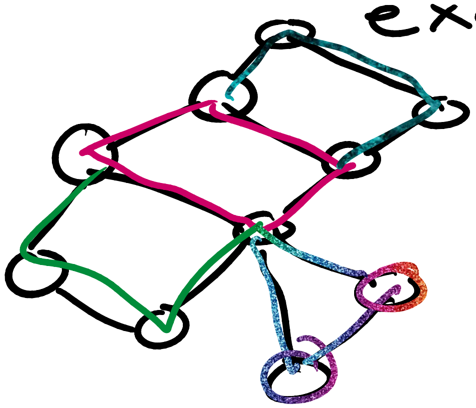
$\Rightarrow$ this builds our open ✓ ear decomposition QEWL

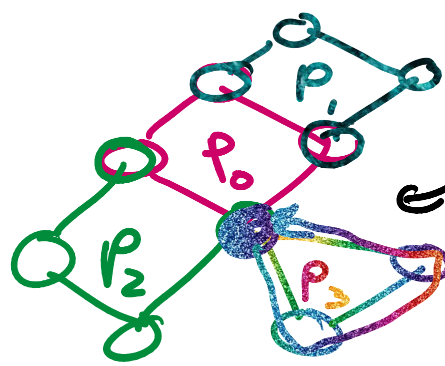
Closed-ear decompositions

$O = P_0 \dots P_k$

P_0 = cycle

P_i = an open OR closed path whose endpoints exist on $P_0 \cup P_1 \cup \dots \cup P_{i-1}$





closed ear

G is 2-edge-connected

$\Leftrightarrow$ G has closed ear decomposition

$\Leftrightarrow \forall u,v \in V(G) : \exists\, 2$ u,v-iedps

$\iff \forall u, v \in V(G): \exists\, 2\ u,v$-paths

(internally edge-disjoint paths)

Biconnectivity

→ a biconnected graph has no cut vertices

☆ → K_1 and K_2 are biconnected ☆

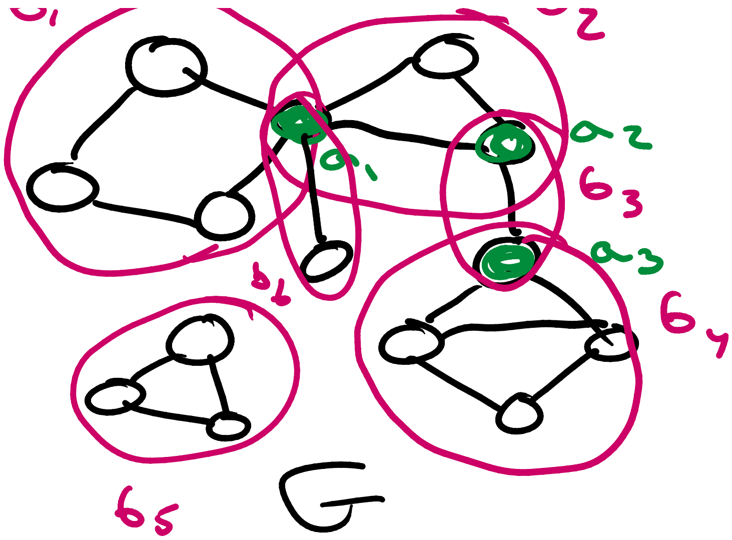
Block decomposition of G

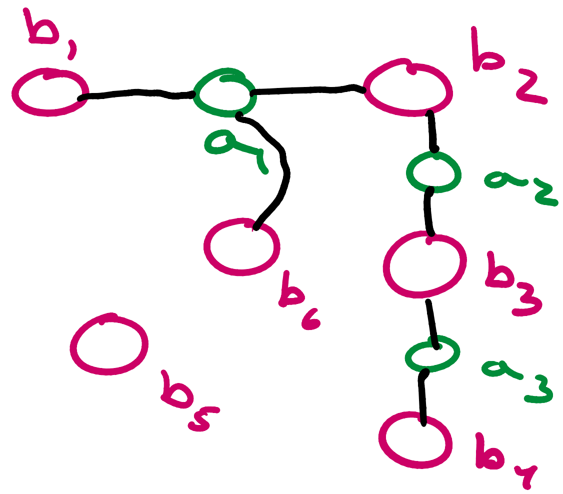
- <u>Blocks</u> are maximal biconnected <u>subgraphs</u> of G

  AKA biconnected components (BiCCs)

- <u>Articulation vertices</u> are cut vertices of G

Using our blocks and cut vertices, we can construct a block-cutpoint bipartite graph


b1
b2
a1
a2
b3
a3
b4
b6
b5
G


b1
a1
b2
a2
b6
b3
a3
b5
b4