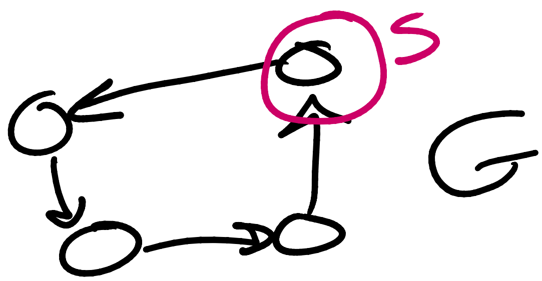
Lecture 13 - k-connectivity

Tuesday, March 9, 2021 2:24 PM

Digraph connectivity

vertex cut - a set $S \subseteq V(G)$ such that $G-S$ is not strongly connected



$K(G)$ = connectivity of digraph G
= minimum size of some S

if $k \leq K(G) \Rightarrow G$ is k-connected

edge cut - a set $F \subseteq E(G)$ that separates $V(G)$ into two set $S, \bar{S}$

the size of this cut is the number of edges from $S \to \bar{S}$

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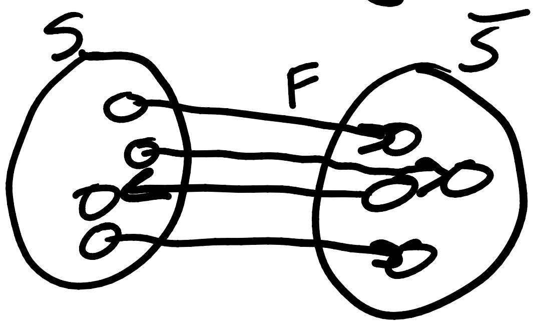
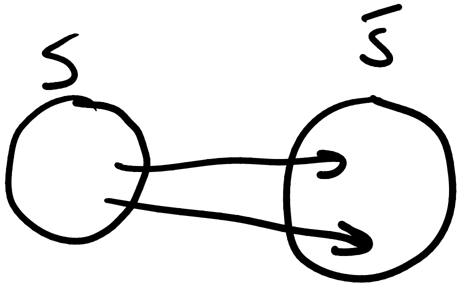


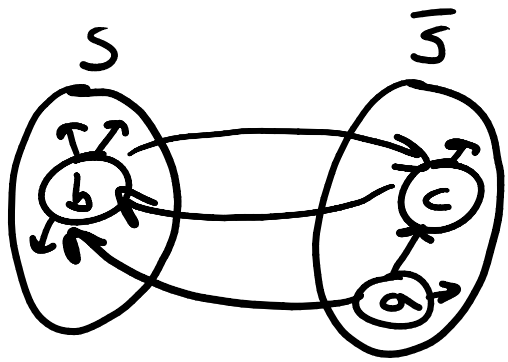
size of cut = 3

$K'(G)$ = edge-connectivity of G

= minimum size of a cut



cut = 2



k = 2? ✗

k = 1 ✓

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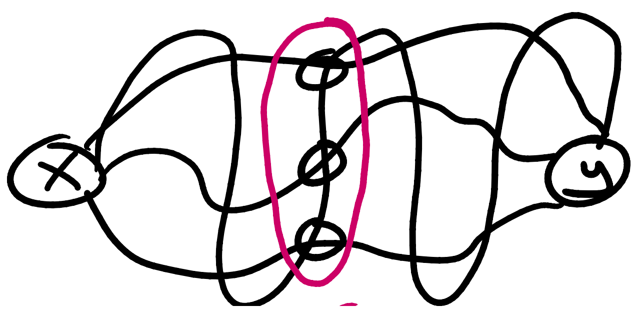
k-connectivity

x, y-separator - a set $S \subseteq V(G)$ s.t. $G - S$ has no x, y-paths



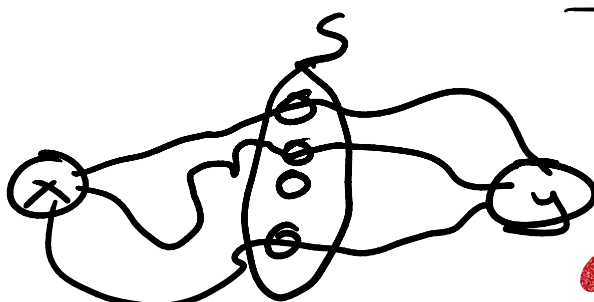
$K(G)$ = minimum x,y-separator over all $x, y \in V(G)$

over all $x,y \in V(G)$

$K(x,y)$ = connectivity of x,y
= minimum size of x,y-separator

$\lambda(x,y)$ = maximum number of internally disjoint x,y-paths

as every x,y-separator must contain a vertex from each internally disjoint path $\Rightarrow$ $\underline{K(x,y) \geq \lambda(x,y)}$



?Big Question?

does $K(x,y) = \lambda(x,y)$?

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Menger: it does: $K(x,y) = \lambda(x,y)$
if $(x,y) \notin E(G)$

let's use the power of strong  induction to prove this

induction on $|V(G)|$

Basis $|V(G)| = 2$ ⓧ ⓨ

$\lambda(x,y) = 0$, $K(x,y) = 0$ ✓

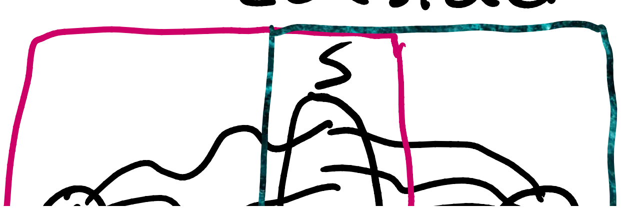
Assume we have G w/ $|V(G)| = n$

- assume we have $K(x,y) = k = |S|$

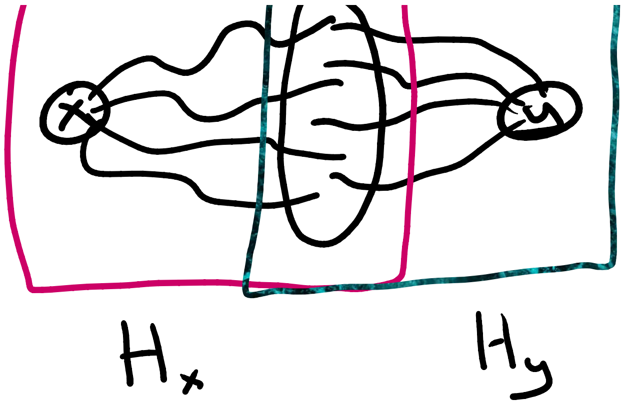
→ construct k idps given min cut S

Case 1: $S \neq N(y)$ or $N(x)$

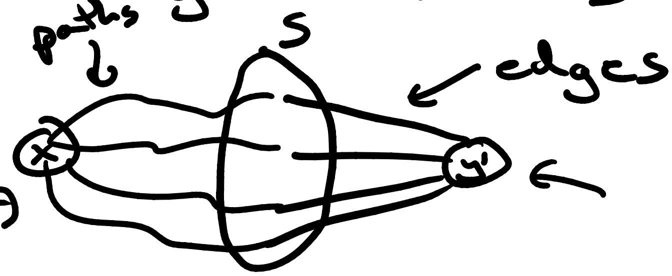
- consider x, S-paths
- consider Y, S-paths



- define graph
  $H' = H$ + vertex y'



- define g
$H_1 = H_x +$ vertex y' and edges from y' to all $s \in S$



I.H. gives us k-idps

- define H_2 in the same way but with H_y and an x'

Note: in H_1 and H_2, S is still a minimum separator

Note: $|V(H_1)| < |V(G)|$, $|V(H_2)| < |V(G)|$

$\Rightarrow$ inductive hypothesis on H_1 and H_2 $\rightarrow$ we have k-idps on both

$\Rightarrow$ we can combine these paths to get k-idps on G ✓

Case 2: $S = N(x)$ and/or $N(y)$

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2a) $\exists v \notin \{x\} \cup \{y\} \cup N(x) \cup N(y)$

- consider $G - v$
- I.H. on $G - v$

Note: v is not on minimum cut

$\Rightarrow$ k-idps on $G - v \Rightarrow$ k-idps on G ✓

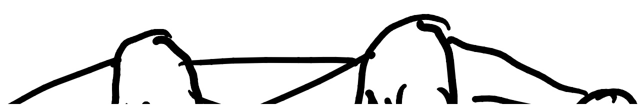
2b) if $\exists v \in N(x) \cap N(y)$

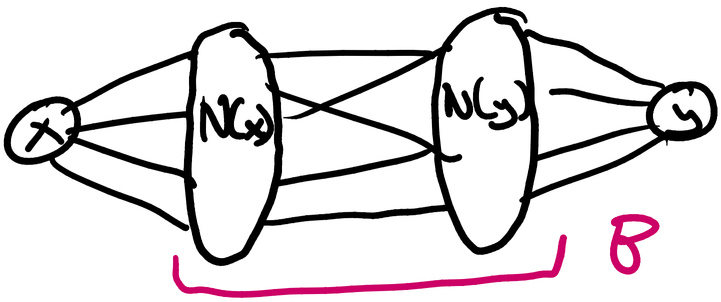
- consider $G - v$
- I.H. on $G - v \Rightarrow (k-1)$ idps

$\Rightarrow$ to get back to G, we have path (x, v, y) to give us k idps on G ✓

2c) Otherwise, both $N(y)$ and $N(x)$ are separators

$\leftarrow$ only possible configuration



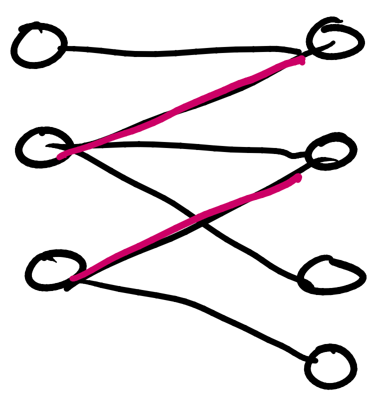


← only po… configuration after 2a & 2b

- define bipartite graph B as N(x), N(y), and the edges between them

- Note: every x,y-path uses an edge in B

Can we guarantee that there will be k-idps using k edges in B?



← Meganote: this is reduced to a matching problem

Ultranote: as each of N(x) and N(y) are separtors, they are also vertex covers in B

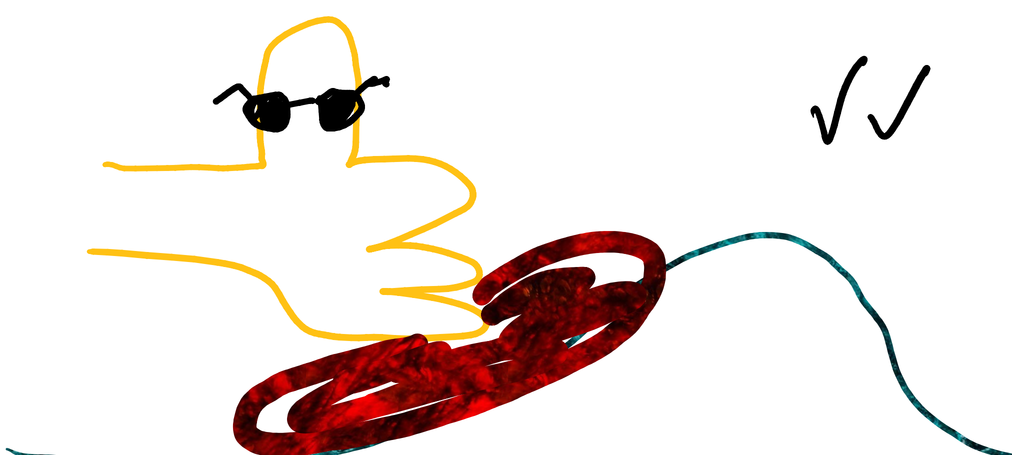
→ minimum cover is going to be bounded below by the minimum of $|N(x)|$ or $|N(y)| = |S|$

⇒ from König-Egerváry, min cover = max match

⇒ so we have a max match of size $|S|$

⇒ w have $|S| = k$ idps ✓

QEWL



✓✓

Same stuff but different for edge connectivity edge-connectivity

$K'(x,y)$ = x,y-edge-connectivity

= minimum x,y edge cut

$\lambda'(x,y)$ = maximum number of edge-disjoint paths

$$K'(x,y) = \lambda'(x,y)$$

G is k-edge-connected if

$$\forall\, x,y \in V(G) : K'(x,y) \geq k$$
$$\lambda'(x,y) \geq k$$