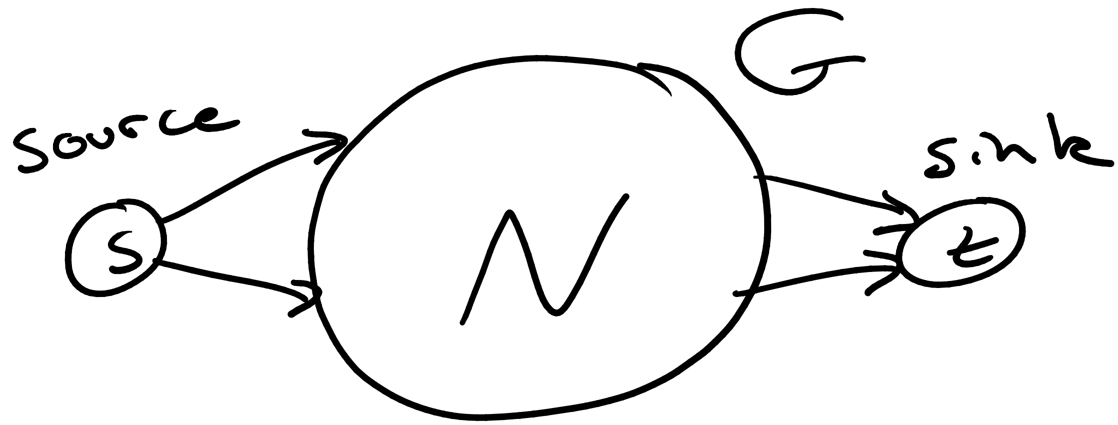
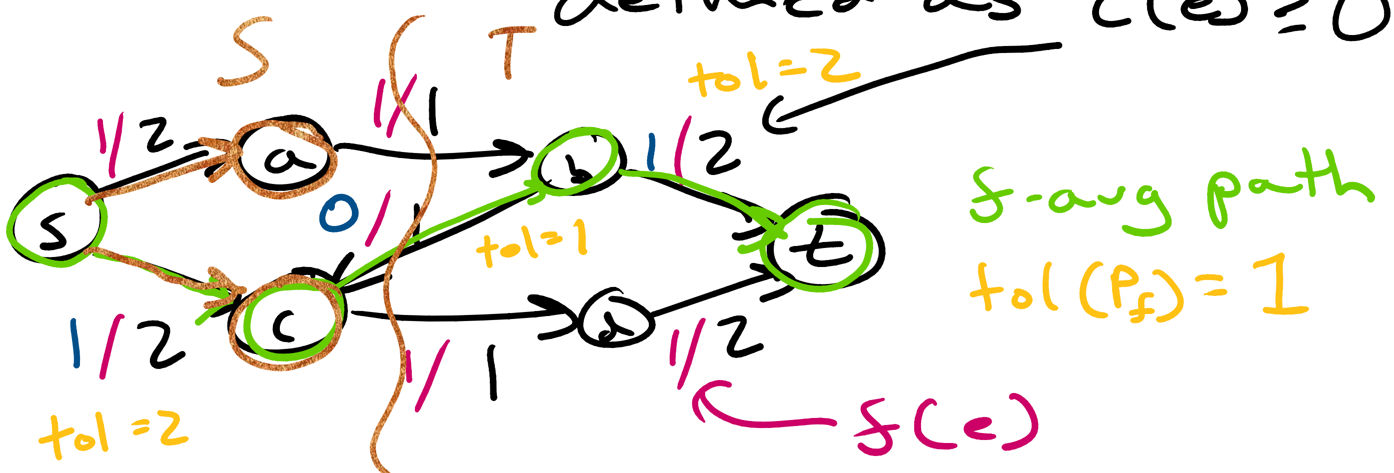
Flow network



$\forall e \in E(G)$: we have a capacity defined as $c(e) \geq 0$



a flow on G assigns to each edge a flow value f(e)

feasible flow values for e

$0 \leq f(e) \leq c(e)$

feasible flows for $v \in V(G)$, $v \neq s, t$

$f^-(v)$ = flow into v

$f^+(v)$ = flow out of v

$f^+(v) = f^-(v)$

flow value of our whole G

$val(f)$ = total flow

↑ defines our current flow

$val(f) = f^+(s) - f^-(s)$ ← usually zero

$= f^-(t) - f^+(t)$

maximum flow = feasible flow where $val(f)$ is maximum

given feasible flow f, an f-augmenting path is a source → sink where $\forall e \in P_f$:

source $\rightarrow$ sink where $\forall e \in P_f$:

- if P_f follows direction of e then $f(e) < c(e)$
- if P_f doesn't follow direction of e then $f(e) > 0$

$\epsilon(e) = c(e) - f(e) =$ tolerance of e for forward edge

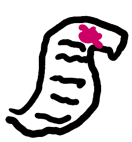
$\epsilon(e) = f(e) =$ tolerance of e for backwards edge

Given P_f, we consider the minimum tolerance over all $e \in P_f = z$

To augment our flow

$\forall e \in P_f$: $f(e) + z$ if we follow e forward

$f(e) - z$ if we follow e backward

Note  : when we augment a flow, Note : when we augment a flow, we increase val(f) by z

Source-sink cut $[S, T]$

S = source set of vertices

T = sink set of vertices

note: $s \in S$, $t \in T$

the size of the cut aka the capacity of the cut

$= \sum c(e) \ \forall e \in [S, T]$

Note: the size of a cut gives us a bound on a network's flow

$$|[S,T]| \geq val(f)$$

? Min cut = max flow ?

Answer = yes

Let's show some equivalences

1. f is a max flow

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2. no f-augmenting path

3. $|[S,T]| = \text{val}(f)$

we'll show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

$1 \Rightarrow 2$ Contrapositive

$\neg 2 \Rightarrow \neg 1$

$\exists$ f-aug path $\Rightarrow$ f is not max flow

we've already seen how to use an f-aug path to increas the flow on a network ✓

$2 \Rightarrow 3$

no f-aug path $\Rightarrow$ cut equal to flow on the network

S = set of reachable vertices

note: $s \in S$, $t \notin S$

all edges from $S \to T$ have

$c(e) = f(e)$

all edges from $T \to S$ have

$f(e) = 0$

$$\mathrm{val}(f) = \sum \text{flows from } S \to T$$
$$- \sum \text{flows from } T \to S$$
$$= \sum \text{flows from } S \to T$$
$$= \sum_{e \in [S,T]} c(e) = |[S,T]|$$
$$\Rightarrow \mathrm{val}(f) = |[S,T]| \checkmark$$

$3 \Rightarrow 1$

cut = flow $\Rightarrow$ flow is max

Note: the capacities on edges gives us our cut = flow

Can we increase this flow?

No → we've seen this before
edges are at capacity

$$\sum_{e \in [S,T]} f(e) = \sum_{e \in [S,T]} c(e)$$

⇒ our flow is maximum ✓

Combined with our inequality
above → cut ≥ max flow
and → cut = flow

$\boxed{\text{minimum cut} = \text{maximum flow}}$

QEWL

To get max-flow/min cut:
initialize all flows to zero
while ∃ some f-aug path:
find min tol on path
use it to update flows

find min for the path

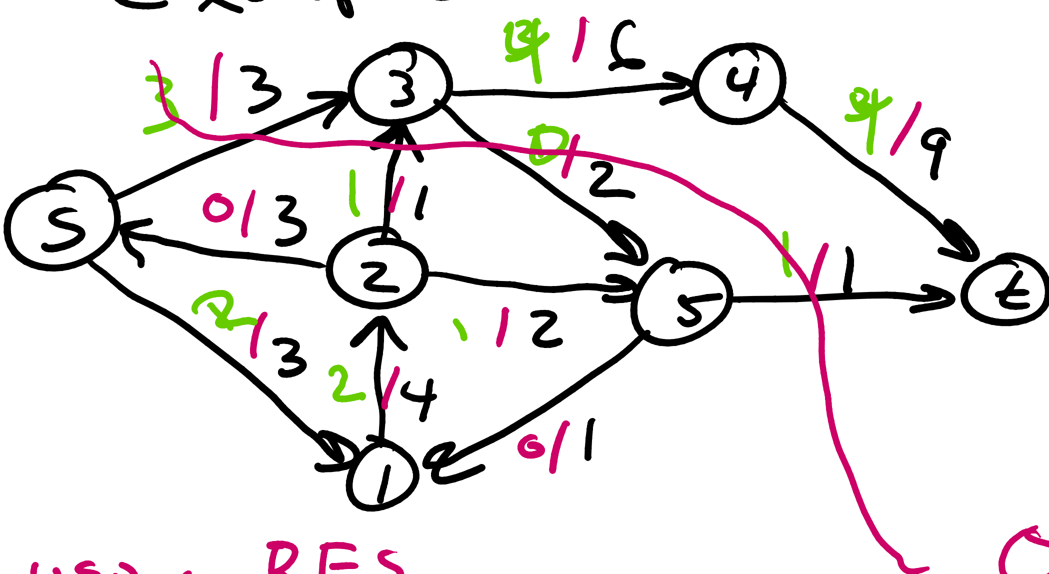
use it to update flows

=> we're done

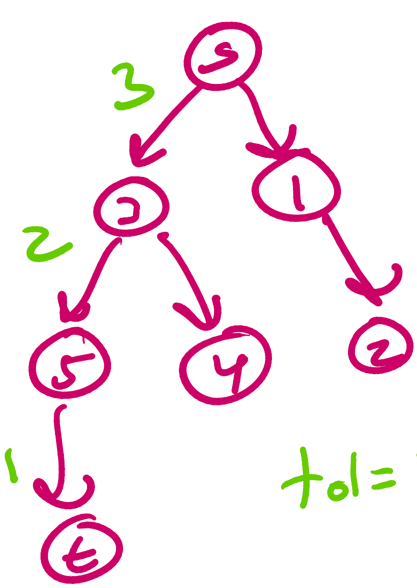
min cut is defined as [S,T],

where S is "reachable" vertices

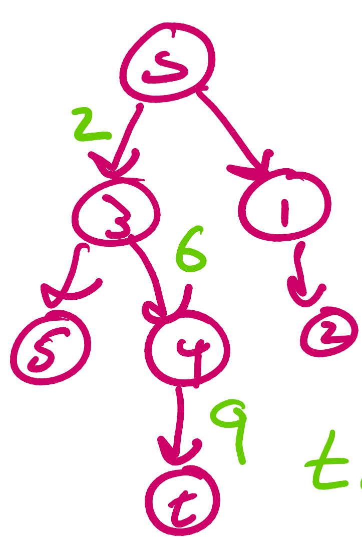
Example



using BFS



tol = 1

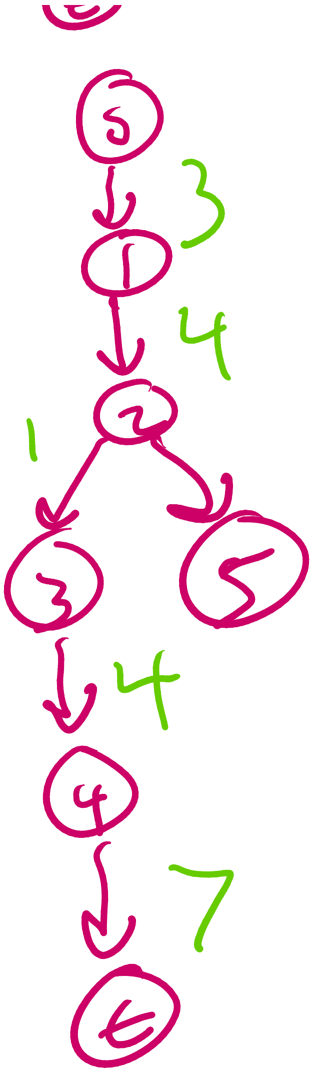


tol = 2

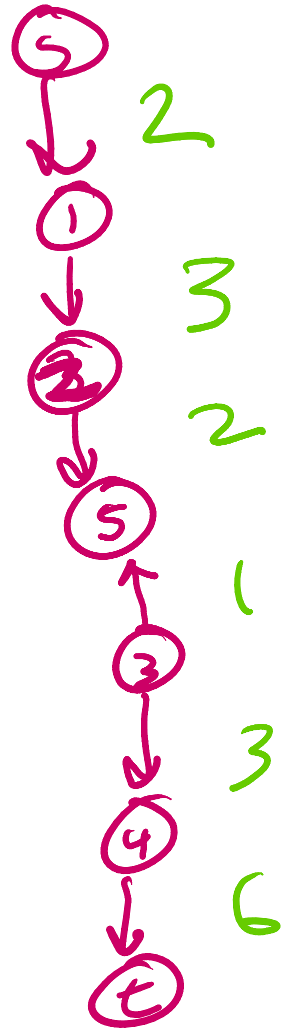
our [S,T]

cut = val(f) = 5

S



tol=1



t=1