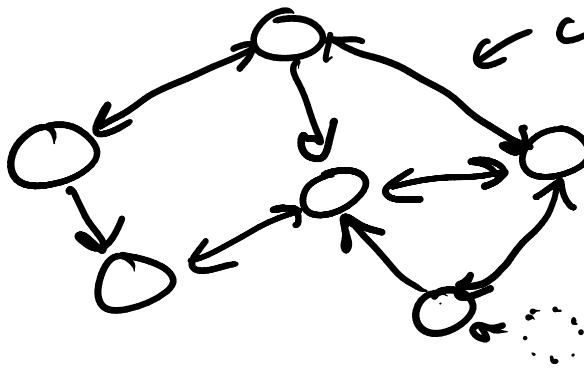
Lecture 18 - Graph Coloring

Friday, March 26, 2021 2:18 PM

Applications

biconnectivity

U.S. military in-field comm. network

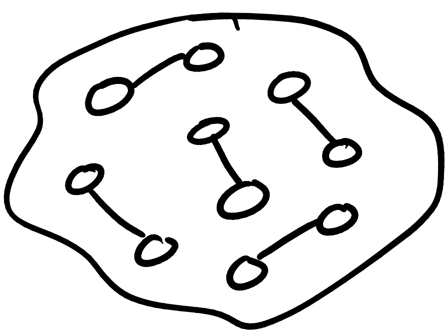


comm. distance is restricted

note: removal here disconnects network

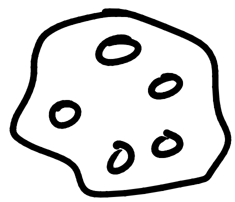
Matching

Graph partitioning



max match

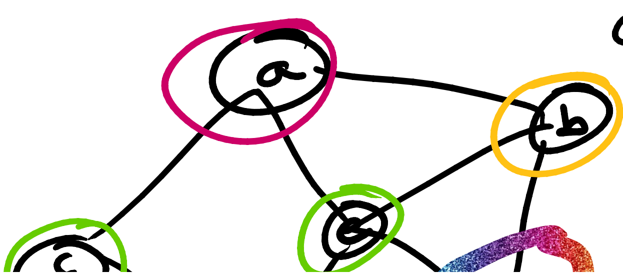
coarsen



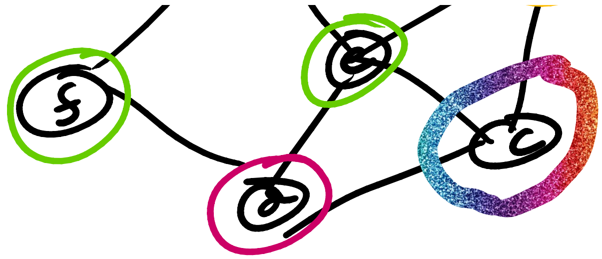
then cut

via edge contraction

Coloring



dependency graph



Graph coloring

k-coloring of G is a labeling $f : V(G) \to S$, $k = |S|$

proper coloring: a k-coloring of G s.t. no neighboring vertices have the same color

G is k-colorable if it can be properly colored w/ k colors

Chromatic number of $G \to \chi(G)$

$\chi(G)$ = the minimum k for which G is k-colorable

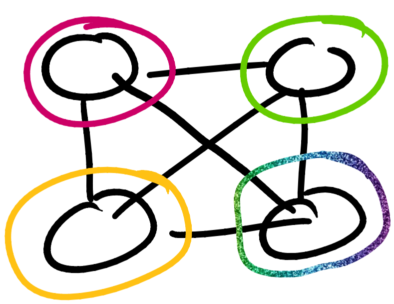
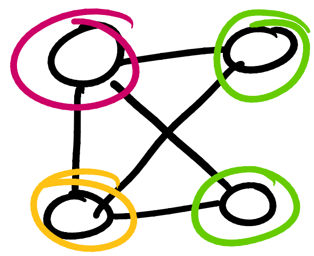
Optmal coloring of G is a k-coloring for $\chi(G) = k$

k-coloring for

G is <u>color-critical</u> if for all subgraphs $H \subseteq G$, $H \not\equiv G$,

$$\chi(H) < \chi(G)$$

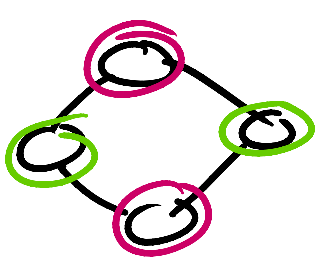
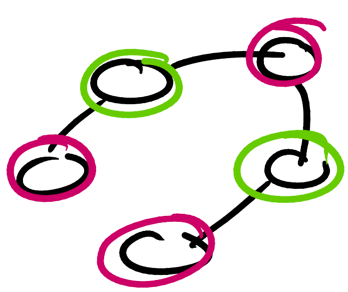
Note: all cliques are color-critical

K_4  

G $\quad$ H

Note: odd cycles are color critical, $\chi(C) = 3$

Note x2: even cycles have $\chi(C) = 2$

Greedy coloring algorithm

Greedy coloring algorithm

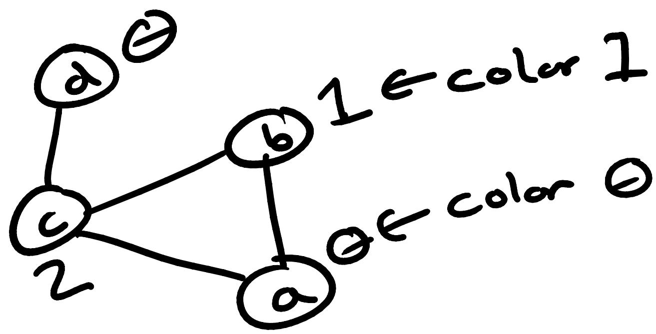
all vertices have empty color

for all vertizes in some order

color vertex with "least"

color doesn't exist in its

neighborhood



Let's talk bounds  -bounds

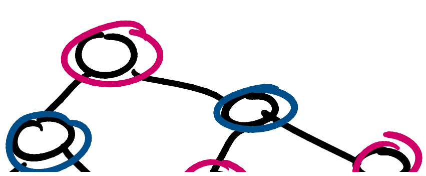
For generic non-null graph G:

$$1 \leq \chi(G) \leq |V(G)|$$

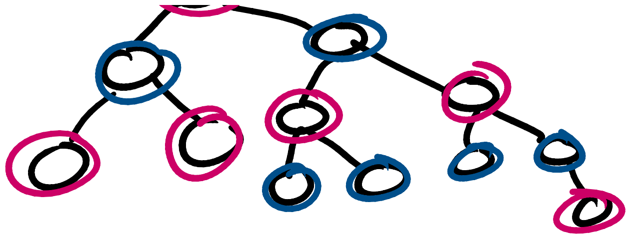
If G is non-empty

$$2 \leq \chi(G)$$

If G is a tree

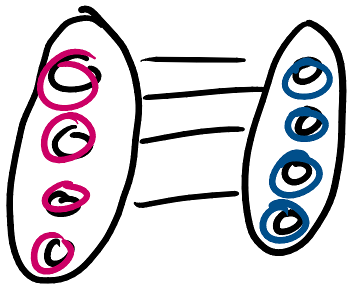


$$\chi(G) = 2$$



$\chi(G) = 2$

If G is bipartite



$\chi(G) = 2$

If G is a clique K_n

$$\chi(G) = |V(G)|$$

If $\alpha(G)$ = the size of the largest independent set of G

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

If $\omega(G)$ = the size of the largest clique in G

$$\chi(G) \geq \omega(G)$$

Considering $\Delta(G)$ and our greedy coloring algorithm greedy coloring algorithm

$$\chi(G) \le \Delta(G) + 1$$

Can we improve on this upper bound?

Brooks says: YES

Brooks: $\chi(G) \le \Delta(G)$

except for odd cycles and cliques

Our approach: construct an ordering for greedy coloring s.t. we can guarantee each vertex we color in order has at most $k-1 = \Delta(G) - 1$ prior colored neighbors

Case 1: G is not k-regular

$(\forall v \in V(G): d(v) = k)$

- consider $u \in V(G): d(u) < \Delta(G)$
- grow a spanning tree from u,

apply order in reverse

apply order in reverse

→ every vertex is guaranteed at least one higher-ordered neighbor

→ order →

Case 2: G is k-regular

- consider $(u, v) \in N(w)$ s.t. $(u, v) \notin E(G)$
  - → can we guarantee that there exists two non-adjacent vertize?
    - → yes, because G is not a clique

- to construct our order:
  - u, v are listed first
  - w is listed last
  - grow spanning tree from w on $G - \{u, v\}$

- grow spanning tree from w on $G - \{u, v\}$

To color → fix $C(u) = C(v)$ ← color of u = color of v

and then process greedy coloring

→ for any vertex at order i, at least $k-1$ lower-ordered neighbors

→ for w, at most $k-1$ colors in $N(w)$

How tite are these bounds?

→ Not very

Note: a tree can have an arbitrarily large max degree

→ but $\chi(T) = 2$ for all T

$$\chi(T) = 2 << \ll < < < < < \Delta(T)$$

What about lower bounds?

$2 \leq \chi(G)$ if G non-empty

$\omega(G) \leq \chi(G)$

Note: if G is triangle-free then $\omega(G) = 2 \leq \chi(G)$

How big can a triangle-free graph be?

Can we come up with a triangle-free graph with an arbitrarily large chromatic number?

→ Mycielski's construction

- Given triangle-free graph G, with $\chi(G) = k$, we can construct triangle-free graph G' with $\chi(G') = k+1$

Consider $v_1 v_2 \dots v_n \in V(G)$

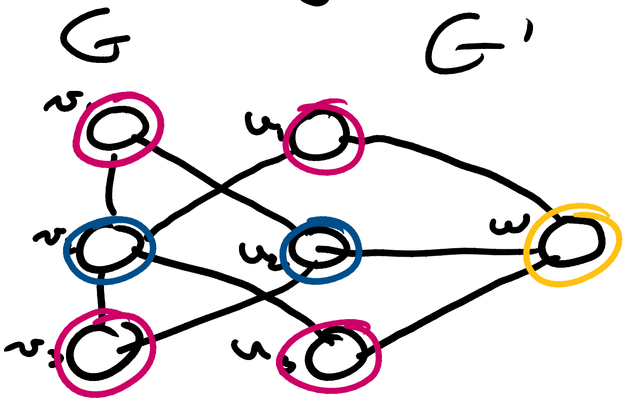
$\updownarrow \quad \updownarrow \quad \updownarrow$

create $u_1 u_2 \dots u_n$

add edges between u_i and all $v_j \in N(v_i)$

create w

add edges from w to all u_i



- no triangles
- $\chi(G') > \chi(G)$

$\omega(G) = 2 <<<<<<<<<<<<<<<<< \chi(G)$